Information Theory

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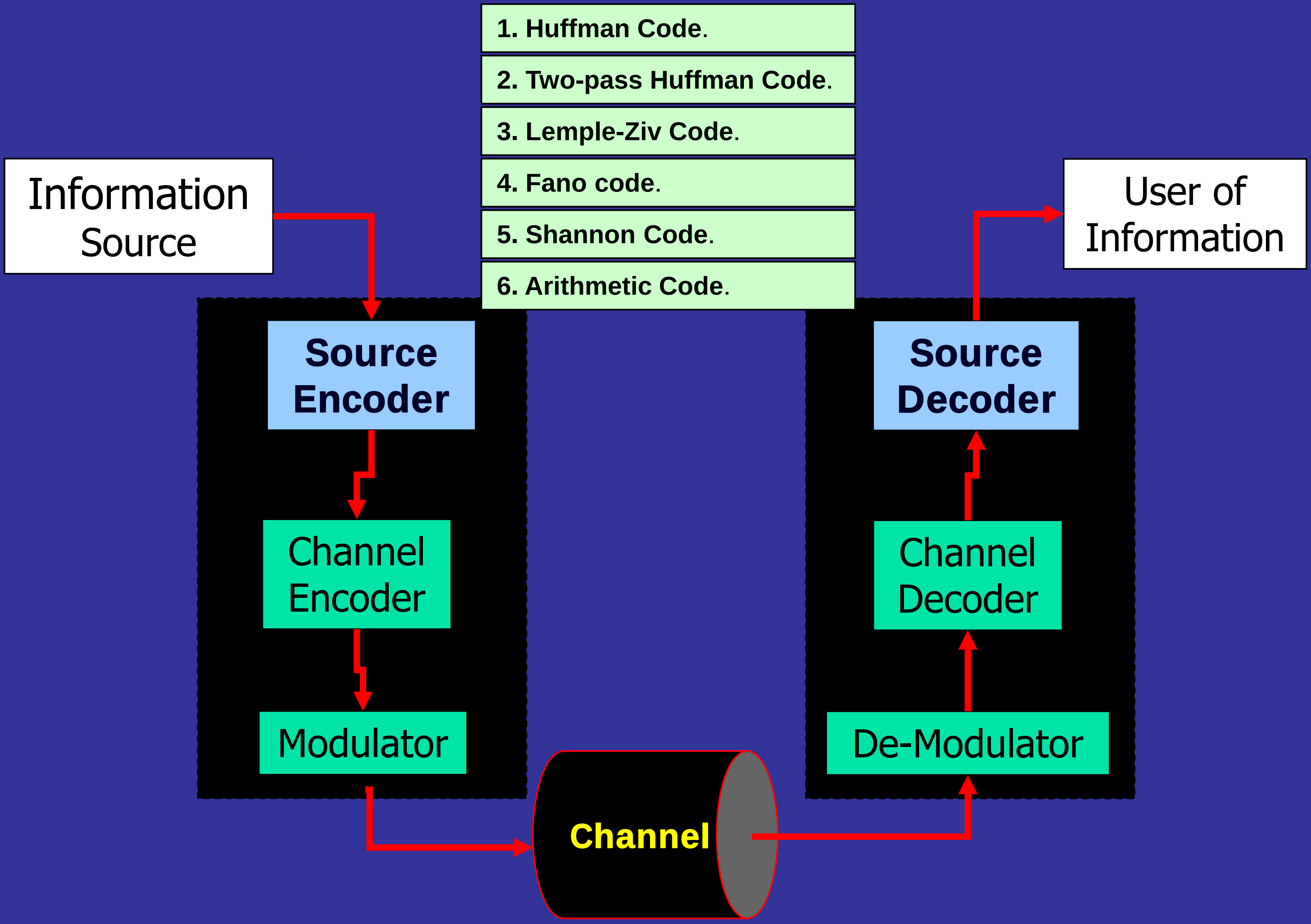
Today's Topics

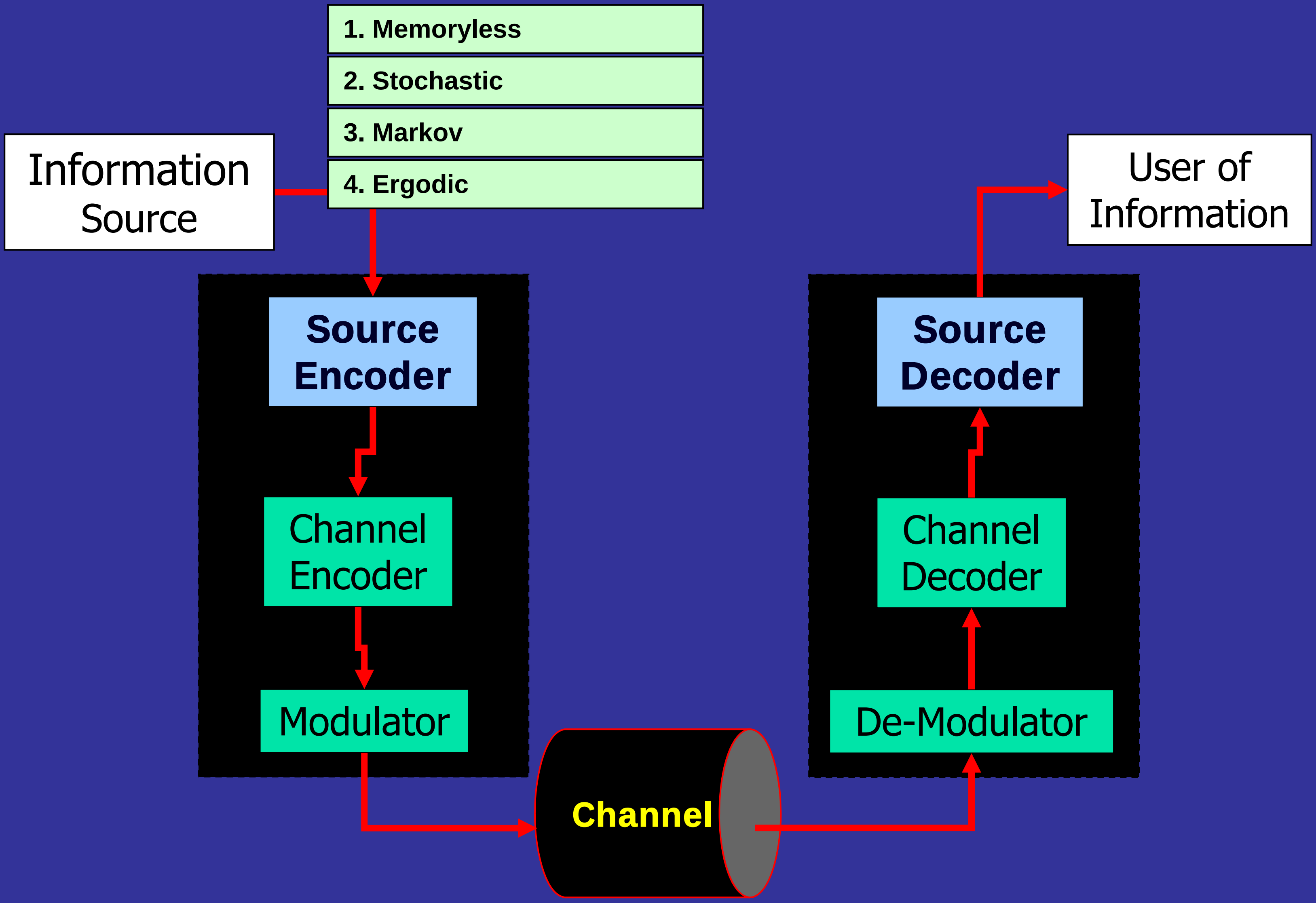
- Channel Coding/Decoding
- Hamming Method:
  - Hamming Distance
  - Hamming Weight
- Hamming (4, 7)

Digital Communication Systems



1. Huffman Code.
2. Two-pass Huffman Code.
3. Lemple-Ziv Code.
4. Fano code.
5. Shannon Code.
6. Arithmetic Code.

Digital Communication Systems


1. Memoryless
2. Stochastic
3. Markov
4. Ergodic

Information Source

User of Information

Source Encoder

Channel Encoder

Modulator

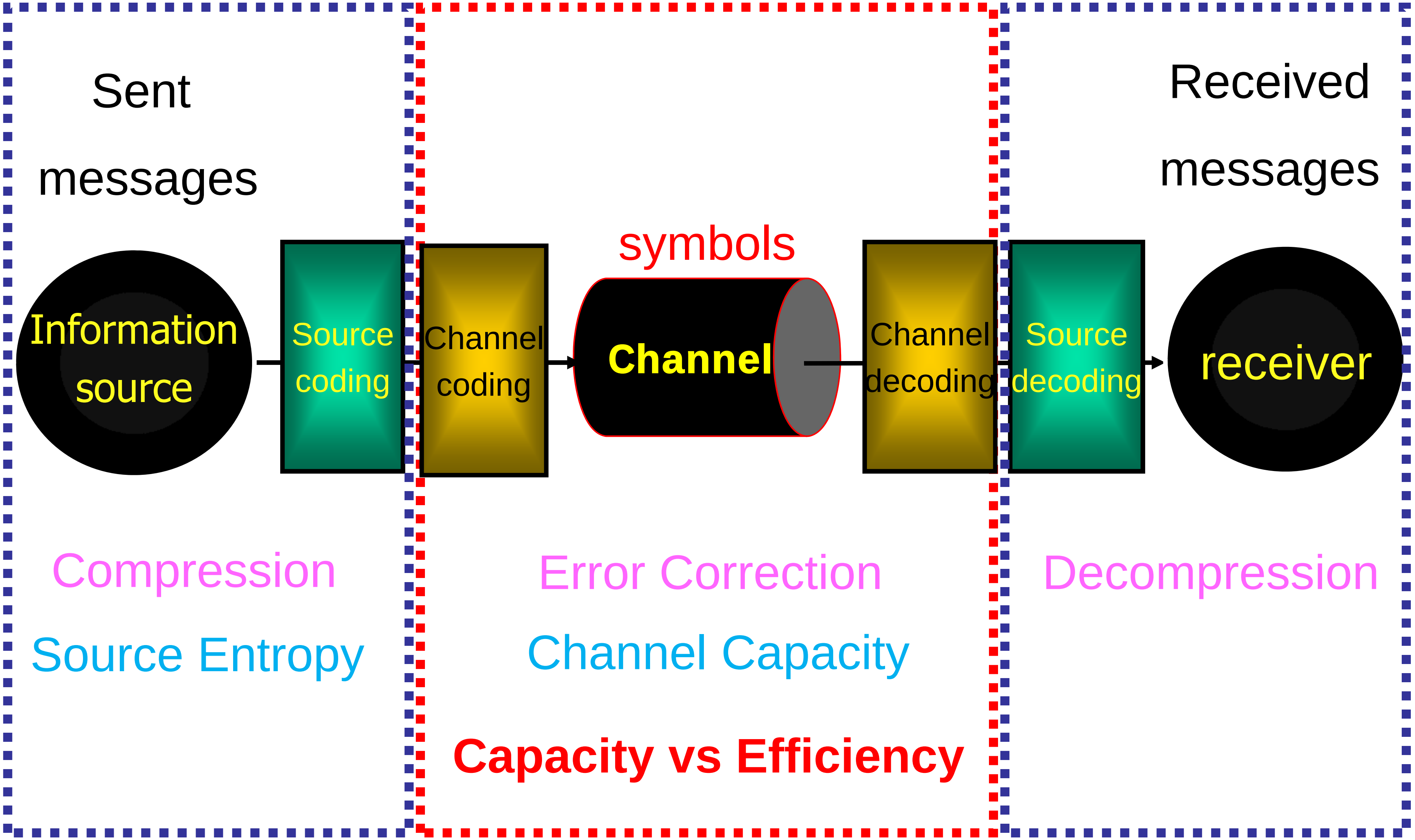
Channel

De-Modulator

Channel Decoder

Source Decoder

INFORMATION TRANSFER ACROSS CHANNELS



Sent messages

Received messages

symbols

Information source

Source coding

Channel coding

Channel

Channel decoding

Source decoding

receiver

Compression

Source Entropy

Error Correction

Channel Capacity

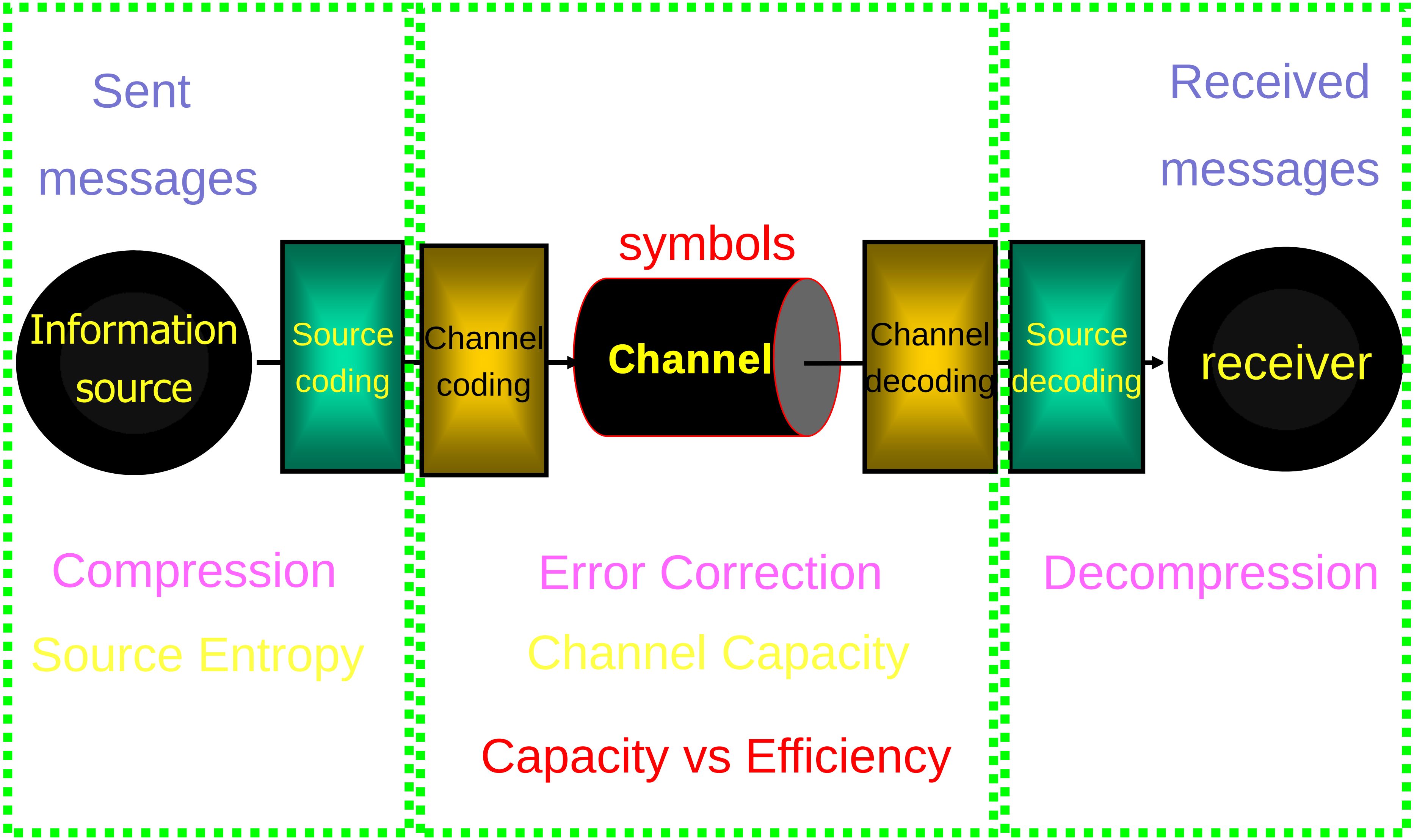
Capacity vs Efficiency

Decompression

Channel Coding/Decoding

Hamming Method

INFORMATION TRANSFER ACROSS CHANNELS



Sent messages

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symbols

Information source

Source coding

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Channel

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Source decoding

receiver

Compression

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Channel Capacity

Capacity vs Efficiency

Decompression

Channel Coding/Decoding

The purpose of channel coding/decoding is to detect and correct errors in noisy channels

Error correction in our life

			
Cell phone	Hard disk	Optical disk	Fiber
10^{-5}–10^{-6}	10^{-14}	10^{-12}	10^{-12}

Channel Coding/Decoding

Error detection and correction

In a noisy channel errors may occur during the transmission of data from information source to destination, so we need a method to detect these errors and then correct them.

Channel Coding/Decoding

Hamming Method

- **It was the first complete error-detecting and error-correcting procedure.**
- **It represents one of the simplest and most common method for the transmission of information (in the presence of noise).**
- **It assumes that the source transmits binary messages (i.e. The information source alphabet is { 0, 1 })**
- **It uses the parity checker method to detect an error**
- **It assumes that the channel is a *binary symmetric channel* (BSC)**

Hamming Codes

- (4,7) Hamming Code detects all one- and two-bit errors
- Corrects all 1-bit errors
- Magic: Any two different codewords differ in at least 3 places!

0000000	0001011	0010111	0011100
0100110	0101101	0110001	0111010
1000101	1001110	1010010	1011001
1100011	1101000	1110100	1111111

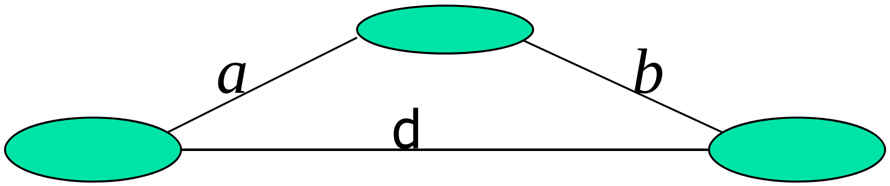
Hamming Distance

- Number of places in which two bit strings differ
-

1	0	0	0	1	0	1
1	0	0	1	1	1	0

= Hamming distance 3

- Acts like a distance:



d $\leq a+b$

Definitions

- Hamming distance between $\underline{x}$ and $\underline{y}$ is

  $d_H := d(\underline{x}, \underline{y})$ is the # of positions where $x_i \neq y_i$

- The minimum distance of a code C is
  - $d_{min} = \min \{ d(\underline{x}, \underline{y}) \mid \underline{x} \in C, \underline{y} \in C, \underline{x}\ \ \underline{y} \}$

- Hamming weight of a vector $\underline{x}$ is
  - $w(\underline{x}) := d(\underline{x}, \underline{0})$ is the # of positions where $x_i \neq 0$

Example

- *Hamming distance* d(1001, 0111) = 3
- *Minimum distance* (101, 011, 110) = 2
- *Hamming weight* w(0110101) = 4

Performance

A code with minimum distance d_{min} is capable of correcting t errors if

$$d_{min} \geq 2t + 1.$$

Proof: If $\leq t$ errors occur, then since $d_{min} \geq 2t + 1$ an incorrect code word has at least t+1 differences with the received word.

Hamming codes

We assume that the sequence of symbols generated by the information source is divided up into blocks of K symbols.

- Minimum distance 3
- Construction

- $G = \begin{pmatrix} I_m & \text{All k-tuples of Hamming weight} > 1 \end{pmatrix}$

- where m = $2^k - k - 1$

I_m is identity matrix

Example: Hamming (7, 4) codes

We assume that the sequence of symbols generated by the information source is divided up into blocks of 4 symbols.

- Generating matrix

  - $G = \begin{pmatrix} I_4 & \mathbf{P} \end{pmatrix}$

P is a 4x3 matrix determined by:

$C_1 = u_2 + u_3 + u_4$

$C_2 = u_1 + u_3 + u_4$

$C_3 = u_1 + u_2 + u_4$

Where + is modulo 2:

0+0=1+1=0 and

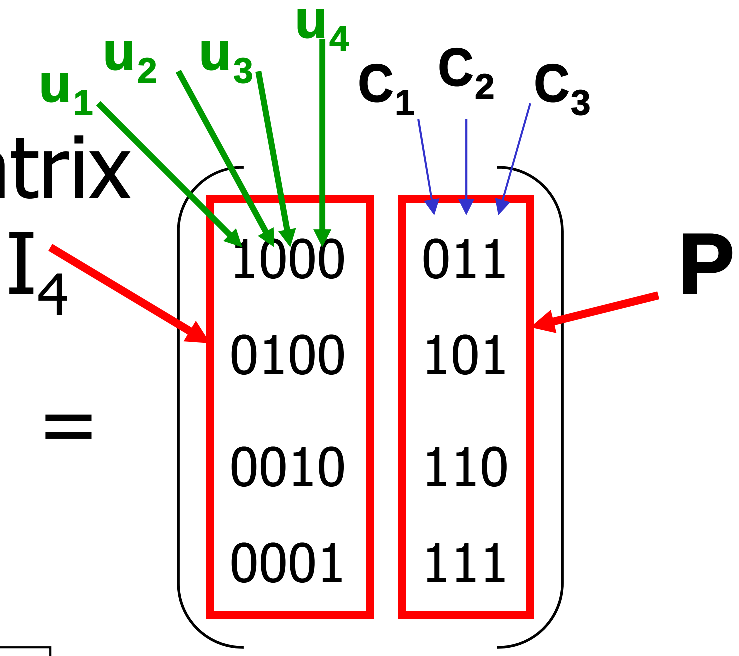
1+0=0+1=1

and u_i are I_4 elements

Example: Hamming (7, 4) codes

We assume that the sequence of symbols generated by the information source is divided up into blocks of 4 symbols. Codewords have length 7

- Generating matrix



$$\bullet\, G = \begin{bmatrix} I_4 & \mathbf{P} \end{bmatrix} = \begin{array}{cc} u_1\,u_2\,u_3\,u_4 & c_1\,c_2\,c_3 \\ \multicolumn{2}{c}{\begin{pmatrix} 1000 & 011 \\ 0100 & 101 \\ 0010 & 110 \\ 0001 & 111 \end{pmatrix}} \end{array}$$

Where + is modulo 2:
$0+0=1+1=0$ and
$1+0=0+1=1$

and u_i are I_4 elements

$$c_1=u_2+u_3+u_4$$
$$c_2=u_1+u_3+u_4$$
$$c_3=u_1+u_2+u_4$$

Hamming (7, 4) Syndrome decoding

H is the parity check matrix

H^T is the Transpose matrix of H

P^T is the Transpose matrix of P

Let $G = [\ I_k\ P\ ]$

For Hamming(7, 4) code: n=7 and k=4

Step 1. construct $H = \begin{pmatrix} P^T & I_{n-k} \end{pmatrix}$

Step 2. Arrange the columns of H in order of increasing binary values

Step 3. Determine the syndrome $S= y.\ H^T$ (y is the received message)

Step 4. If S=0 then no error occurs during transmisstion of information

Step 5. If $S\neq0$ then S gives a binary representation of the error position
(we assume only one error ocuured)

Example:

Suppose that y=(1111011) is received

$$G = \begin{pmatrix} \begin{array}{cc} I_k & P \\ 1000 & 011 \\ 0100 & 101 \\ 0010 & 110 \\ 0001 & 111 \end{array} \end{pmatrix}$$

Step 1

→

$$H = \begin{pmatrix} \begin{array}{cc} P^T & I_{n-k} \\ 0111 & 100 \\ 1011 & 010 \\ 1101 & 001 \end{array} \end{pmatrix}$$

$n=7$ and $k=4$

Step 2

→

$$H = \begin{pmatrix} 0001111 \\ 0110011 \\ 1010101 \end{pmatrix}$$

Step 3

→ $S= y. H^T$ **$= (101) = (5)_{10}$**

Step 5

→ **An error had occurred at position 5 in the received message**

y =(1111011)

→ **The correct sent message is then = (1111111)**