Information Theory

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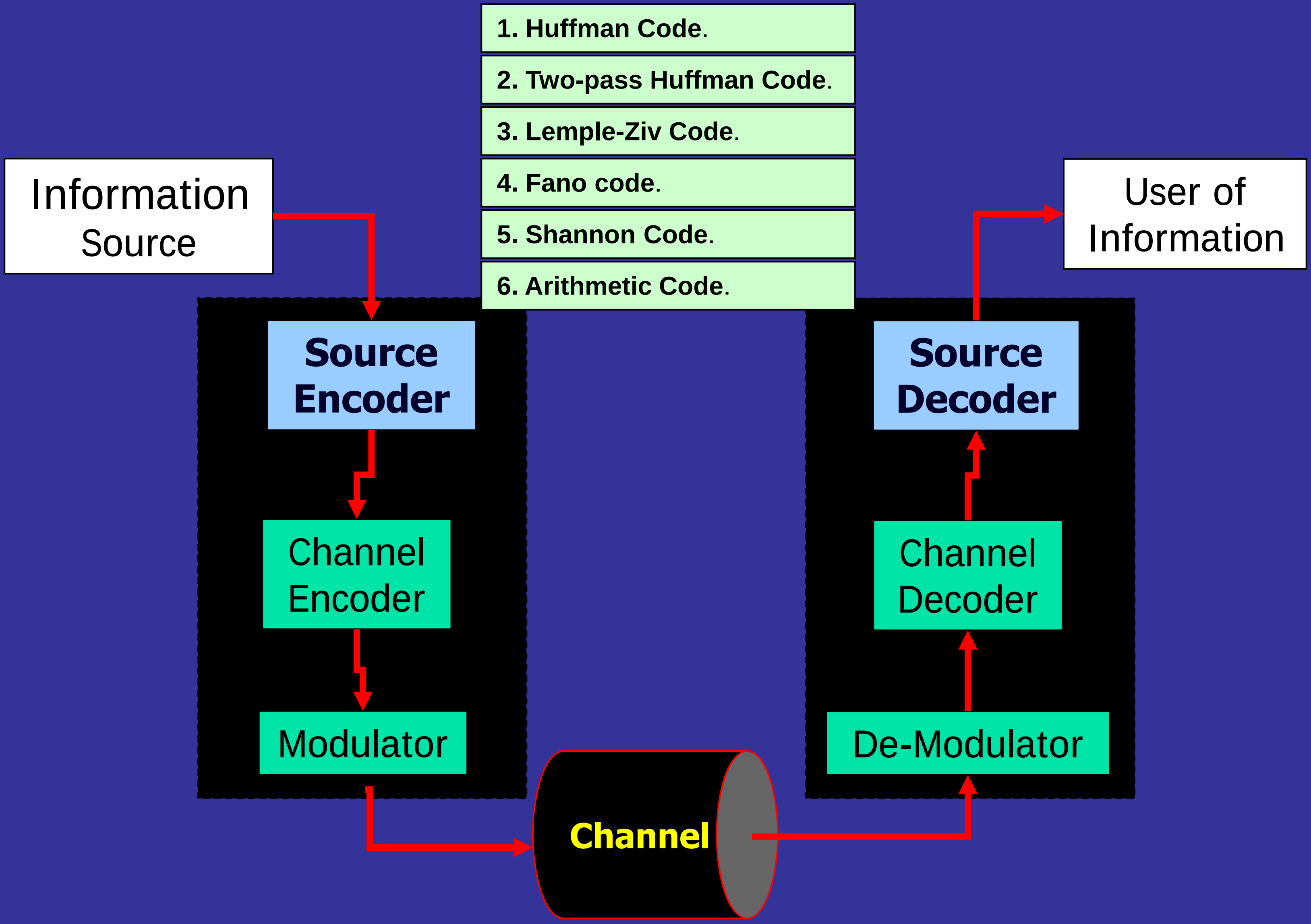
Today's Topics

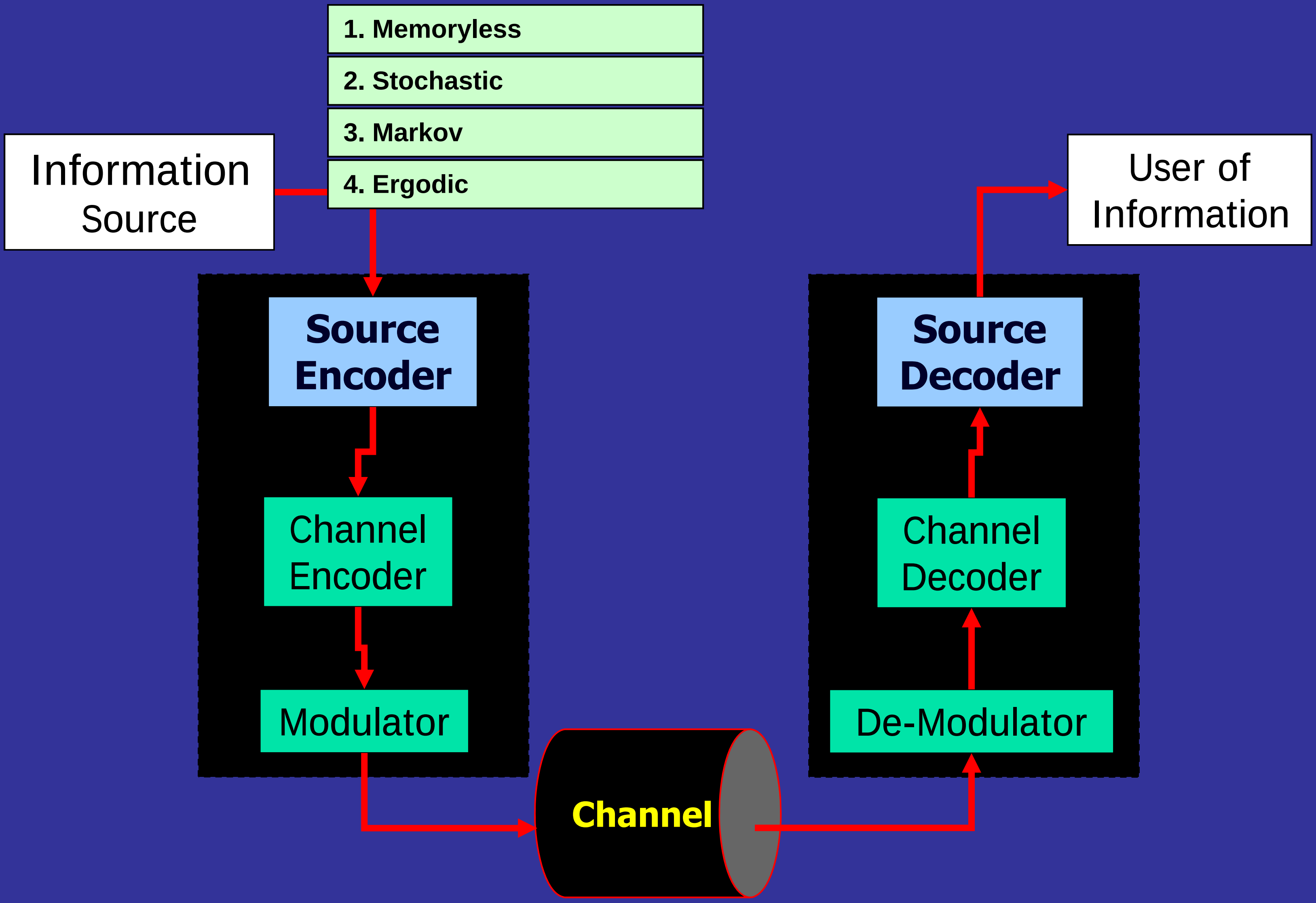
- **Communication Channel**
- **Noiseless binary channel**
- **Binary Symmetric Channel (BSC)**
- **Symmetric Channel**
- **Mutual Information**
- **Channel Capacity**

Digital Communication Systems



1. Huffman Code.
2. Two-pass Huffman Code.
3. Lemple-Ziv Code.
4. Fano code.
5. Shannon Code.
6. Arithmetic Code.

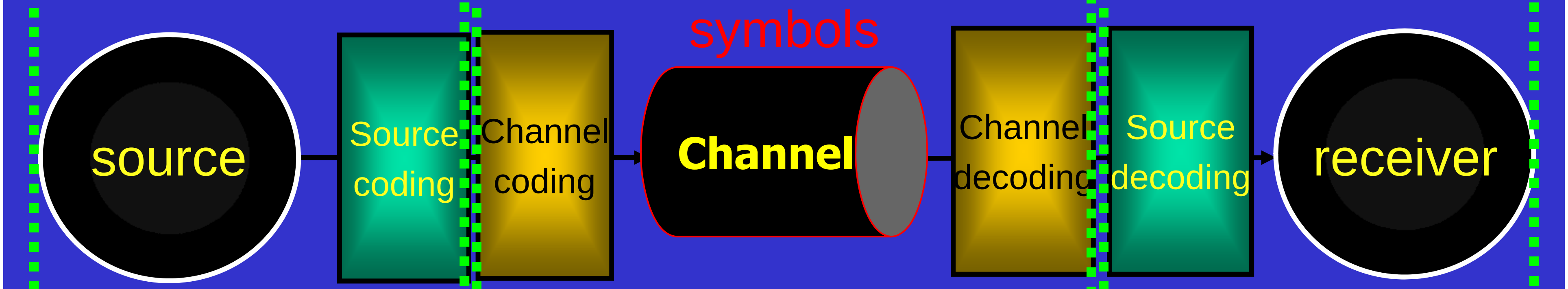
Digital Communication Systems



INFORMATION TRANSFER ACROSS CHANNELS

Sent messages

Received messages



Compression

Error Correction

Decompression

Source Entropy

Channel Capacity

Rate vs Distortion

Capacity vs Efficiency

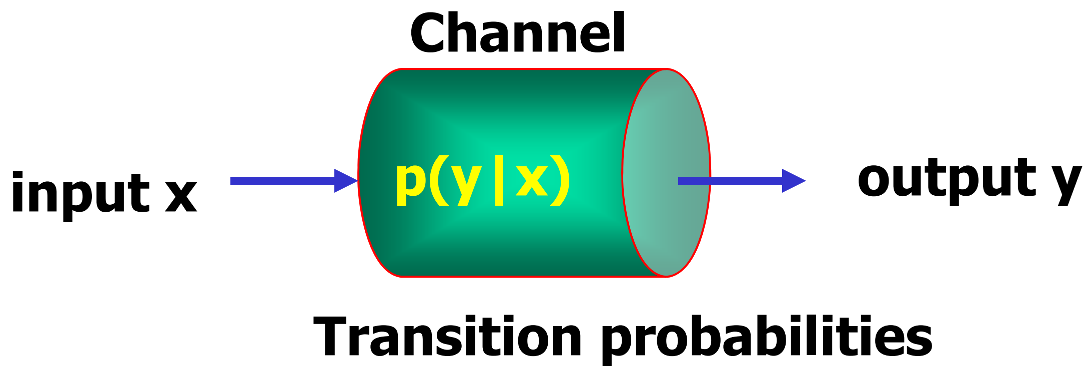
Communication Channel

A (discrete) channel is a system consisting of an input alphabet X and output alphabet Y and a probability transition matrix p(y|x) that expresses the probability of observing the output symbol y given that we send the symbol x

Examples of channels:

CDs, CD – ROMs, DVDs, phones,
Ethernet, Video cassettes etc.

Communication Channel

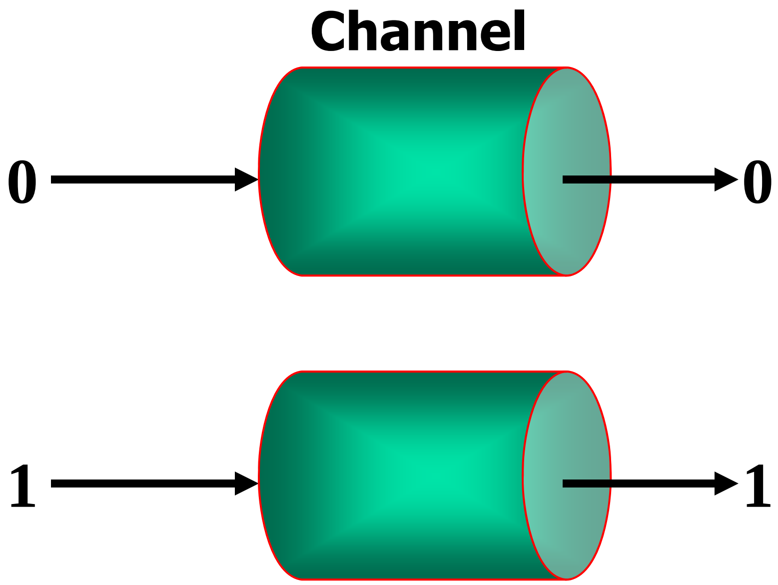


Memoryless:

- output only on input
- input and output alphabet finite

Noiseless binary channel

Noiseless binary channel



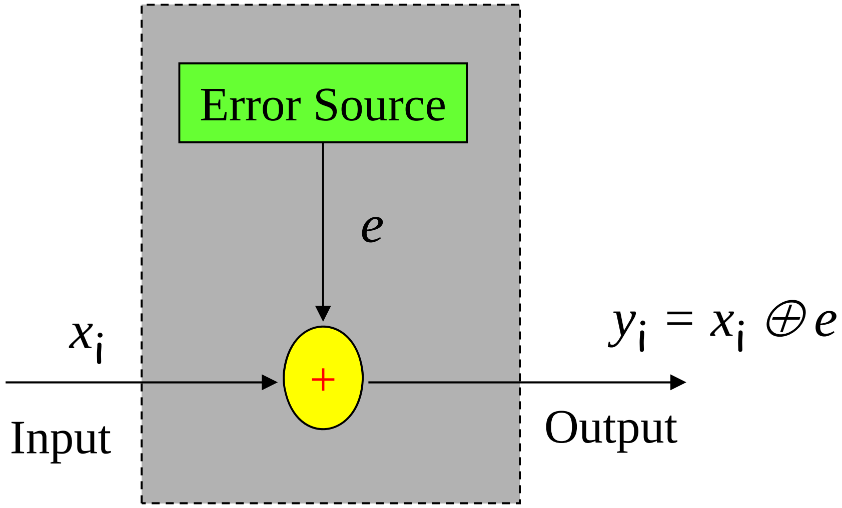
Transition Matrix

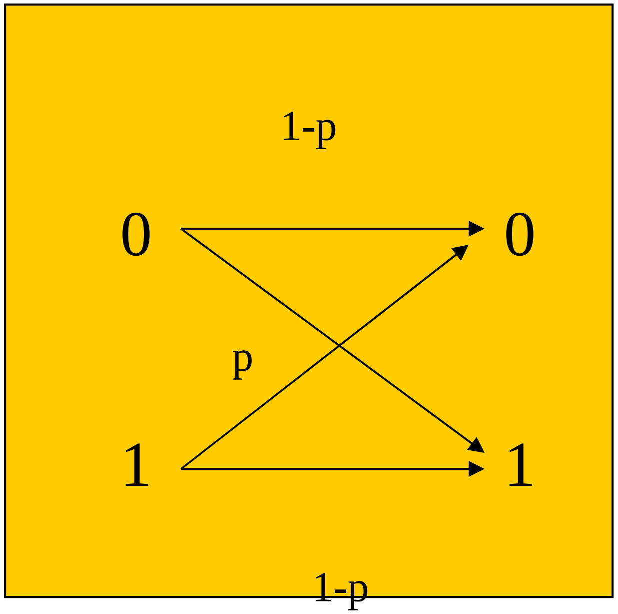
p(y | x) =

	0	1
0	1	0
1	0	1

Binary Symmetric Channel (BSC)

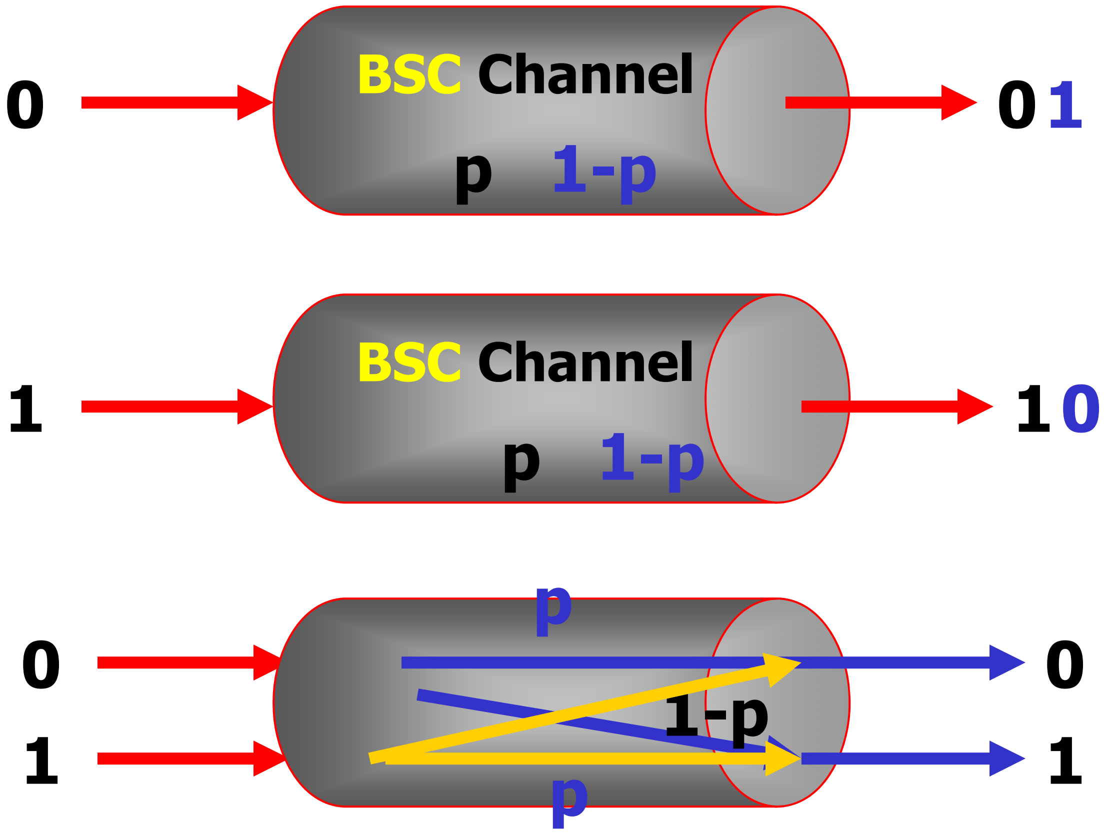
(Noisy channel)





Binary Symmetric Channel (BSC)

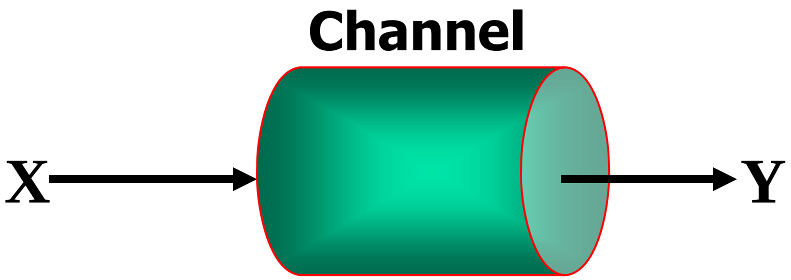
(Noisy channel)



BSC Channel

Symmetric Channel

(Noisy channel)



In the transmission matrix of this channel , all the rows are permutations of each other and so the columns.

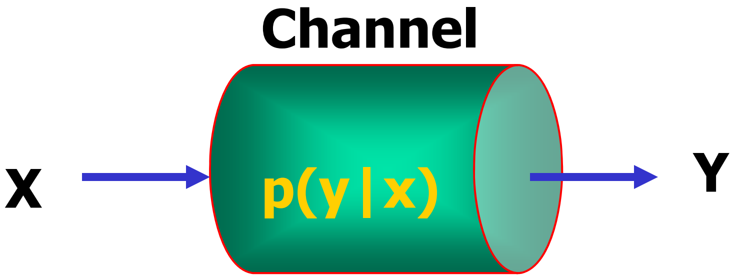
Example:

Transition Matrix

$$p(y \mid x) = \begin{array}{c} \\ x_1 \\ x_2 \\ x_3 \end{array} \begin{array}{c} \begin{array}{ccc} y_1 & y_2 & y_3 \end{array} \\ \begin{pmatrix} 0.3 & 0.2 & 0.5 \\ 0.5 & 0.3 & 0.2 \\ 0.2 & 0.5 & 0.3 \end{pmatrix} \end{array}$$

Mutual Information

Mutual Information (MI)



MI is a measure of the amount of information that one random variable contains about another random variable.

It is the reduction in the uncertainty of one random variable due to the knowledge of the other.

For the two random variable X and Y with a joint probability p(x, y) and marginal probabilities p(x) and p(y).

The mutual information I(X,Y) is the relative entropy between the joint distribution and the product distribution p(x) p(y) i.e.

$$\boxed{I(X,Y) = \sum\sum p(x,y)\log_2\left(\frac{p(x,y)}{p(x)p(y)}\right)}$$

Mutual Information (MI)

Relationship between Entropy and Mutual Information

$$\boxed{I(X,Y)=H(Y)-H(Y|X)}$$

Proof:

$$I(X,Y)=\sum_x\sum_y p(x,y)\log_2(\frac{p(y|x)}{p(y)})$$

$$I(X,Y)=-\sum_x\sum_y p(x,y)\log_2(p(y))+\sum_x\sum_y p(x,y)\log_2(p(y|x))$$

$$I(X,Y)=-\sum_y p(y)\log_2(p(y))+\sum_x p(x)\sum_y p(y|x)\log_2(p(y|x))$$

= H(Y) – H(Y|X)

Note that I(X,Y) ≥ 0

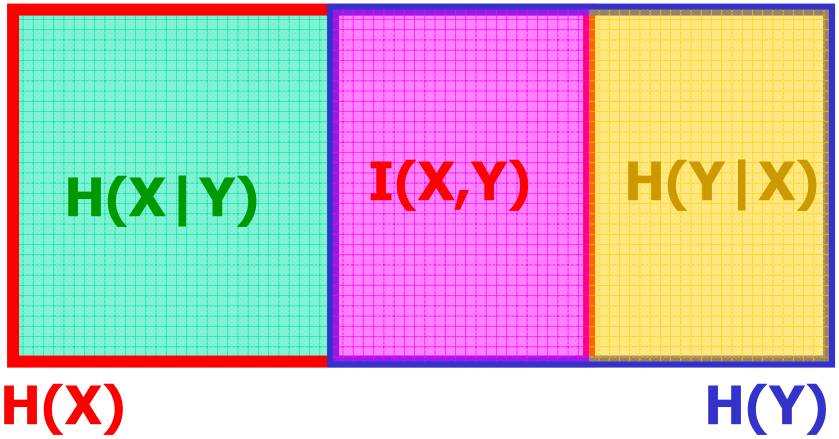
i.e. the **reduction** in the description length of X given Y

or: the amount of information that Y gives about X

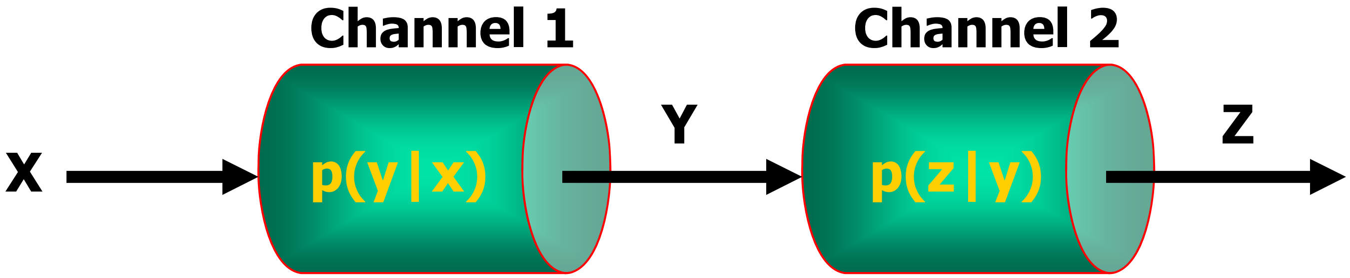
Mutual Information (MI)

Note That:

- $I(X,Y)=H(Y)-H(Y|X)$
- $I(X,Y)=H(X)-H(X|Y)$
- $I(X,Y)=H(X)+H(Y)-H(X,Y)$
- $I(X,Y)=I(Y,X)$
- $I(X,X)=H(X)$



Mutual Information with 2 channels



Let X, Y and Z form a Markov chain: $X \rightarrow Y \rightarrow Z$

and Z is independent from X given Y

i.e. $p(x,y,z) = p(x)\, p(y|x)\, p(z|y)$

The amount of information that Y gives about X given Z is:

$$\boxed{I(X, Y|Z) = H(X|Z) - H(X|YZ)}$$

Mutual Information with 2 channels

Theory: **$I(X,Y) \geq I(X, Z)$**

(I.e. Multi-channels may destroy information)

Proof: $I(X, (Y,Z))$ $=H(Y,Z) - H(Y,Z|X)$

$=H(Y) + H(Z|Y) - H(Y|X) - H(Z|YX)$

$= I(X, Y) + I(X; Z|Y)$

$I(X, (Y,Z))$ $= H(X) - H(X|YZ)$

$= H(X) - H(X|Z) + H(X|Z) - H(X|YZ)$

$= I(X, Z) + I(X,Y|Z)$

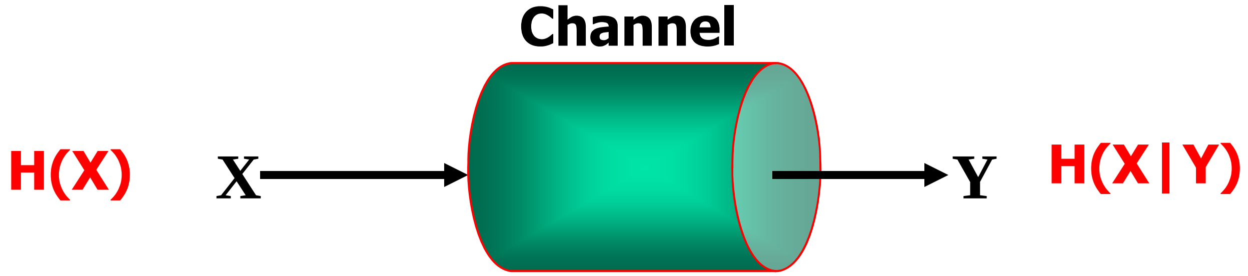
now $I(X, Z|Y) = 0$ (independency)

Thus: $I(X, Y) \geq I(X, Z)$

Channel Capacity

Transmission efficiency

I need on the average H(X) bits/source output to describe the source symbols X After observing Y, I need H(X|Y) bits/source output.



Reduction in description length is called the transmitted Information.

Transmitted R = I(X, Y) = H(X) - H(X|Y)
= H(Y) – H(Y|X) from earlier calculations

We can *<u>maximize R</u>* by changing the input probabilities.

The maximum is called CAPACITY

Channel Capacity

The channel capacity C is the highest rate, in bits per channel use, at which information can be sent with arbitrarily low probability of error

$$C = \max_{p(x)} \; I(X, Y)$$

Where the maximum is taken over all possible input distributions p(x)

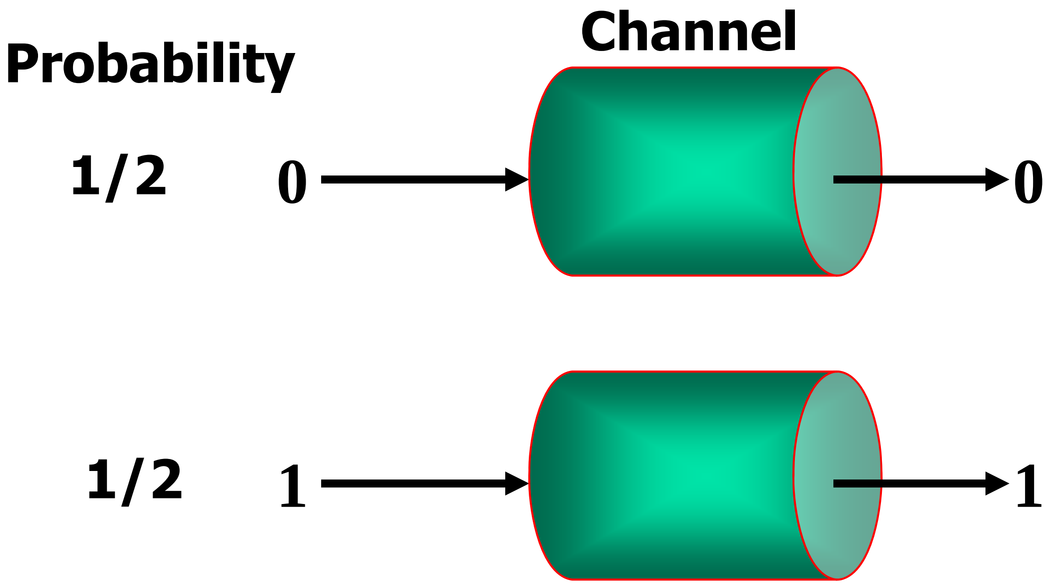
Note that :

- **During data compression, we remove all redundancy in the data to form the most compressed version possible.**
- **During data transmission, we add redundancy in a controlled fashion to combat errors is the channel**

Capacity depends on input probabilities because the transition probabilites are fixed

Channel Capacity

Example 1: **Noiseless binary channel**



Transition Matrix

$$p(y|x) = \begin{array}{|c|c|c|} \hline & 0 & 1 \\ \hline 0 & 1 & 0 \\ \hline 1 & 0 & 1 \\ \hline \end{array}$$

Here the entropy is:

H(X) = H(p, 1-p)= H(1/2,1/2) = -1/2 log 1/2 - 1/2 log 1/2 = 1 bit

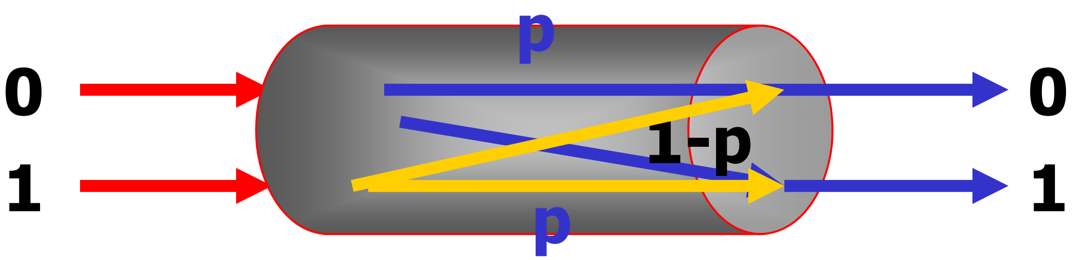
H(X|Y) = -1 log 1 - 0 log 0 - 0 log 0 - 1 log 1 = 0 bit

I(X, Y) = H(X) - H(X | Y) = 1 - 0 = 1 bit

Channel Capacity

Example 2: **(Noisy) Binary symmetric channel**

If P is the probability of an error (noise)



BSC Channel

Transition Matrix

p(y|x)=

	0	1
0	p	1-p
1	1-p	p

Here the mutual information

$$I(X,Y) = H(Y) - H(Y \mid X) \quad = H(Y) - \sum_x p(x)\, H(Y \mid X=x)$$

$$= H(Y) - \sum_x p(x)\, H(P) \quad = H(Y) - H(P) \sum_x p(x)$$

(Note that $\sum_x p(x) = 1$)

$$= H(Y) - H(P)$$

(Note that $H(Y) \leq 1$)

$$\leq 1 - H(P)$$

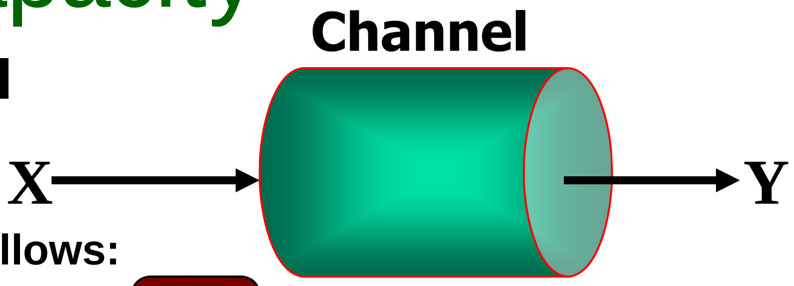
$$\boxed{I(X,Y) \leq 1 - H(P)}$$

Hence the information capacity of the BSC channel is

$$\boxed{C = 1 - H\ (P)\ \text{bits}}$$

(maximum information)

Channel Capacity



Example 3: **(Noisy) Symmetric channel**

The capacity of such channel can be found as follows:

let r be a row of the transition matrix

$$\begin{aligned} I(X, Y) &= H(Y) - H(Y \mid X) = H(Y) - \sum_x p(x)\, H(Y \mid X = x) \\ &= H(Y) - \sum_x p(x)\, H(r) = H(Y) - H(r) \overbrace{\sum_x p(x)}^{=1} = H(Y) - H(r) \\ &\le \log |\mathcal{Y}| - H(r) \end{aligned}$$

$\mathcal{Y}$ is the output alphabet

Hence the capacity C of such channel is: $\boxed{C = \max I(X; Y) = \log |\mathcal{Y}| - H(r)}$

For example: Consider a symmetric channel with the following transmission matrix

It is clear that such channel is symmetric since rows (and columns) are permutations.

$$P(y|x) = \begin{array}{c|ccc} & y_1 & y_2 & y_3 \\ \hline x_1 & 0.3 & 0.2 & 0.5 \\ x_2 & 0.5 & 0.3 & 0.2 \\ x_3 & 0.2 & 0.5 & 0.3 \end{array}$$

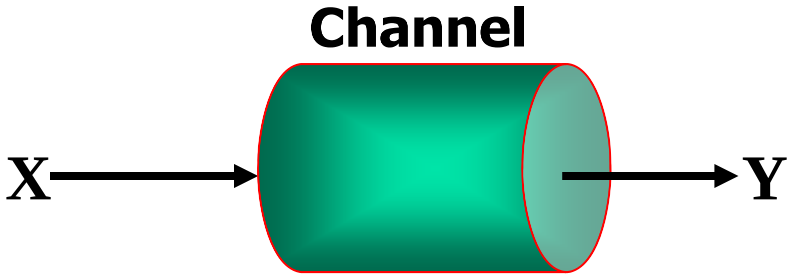
Transition Matrix

Hence the capacity C of this channel is:

C = log 3 - H(0.2, 0.3, 0.5) = 0.1 bits (approx.)

Channel Capacity

(Noisy) Symmetric channel



Note that: **In general for symmetric channel the capacity C is**

$$\boxed{C = \log |\mathcal{Y}| - H(\mathbf{r})}$$

Where $\mathcal{Y}$ **is the output alphabet and r is any row in the transmission matrix.**