Automata and Languages

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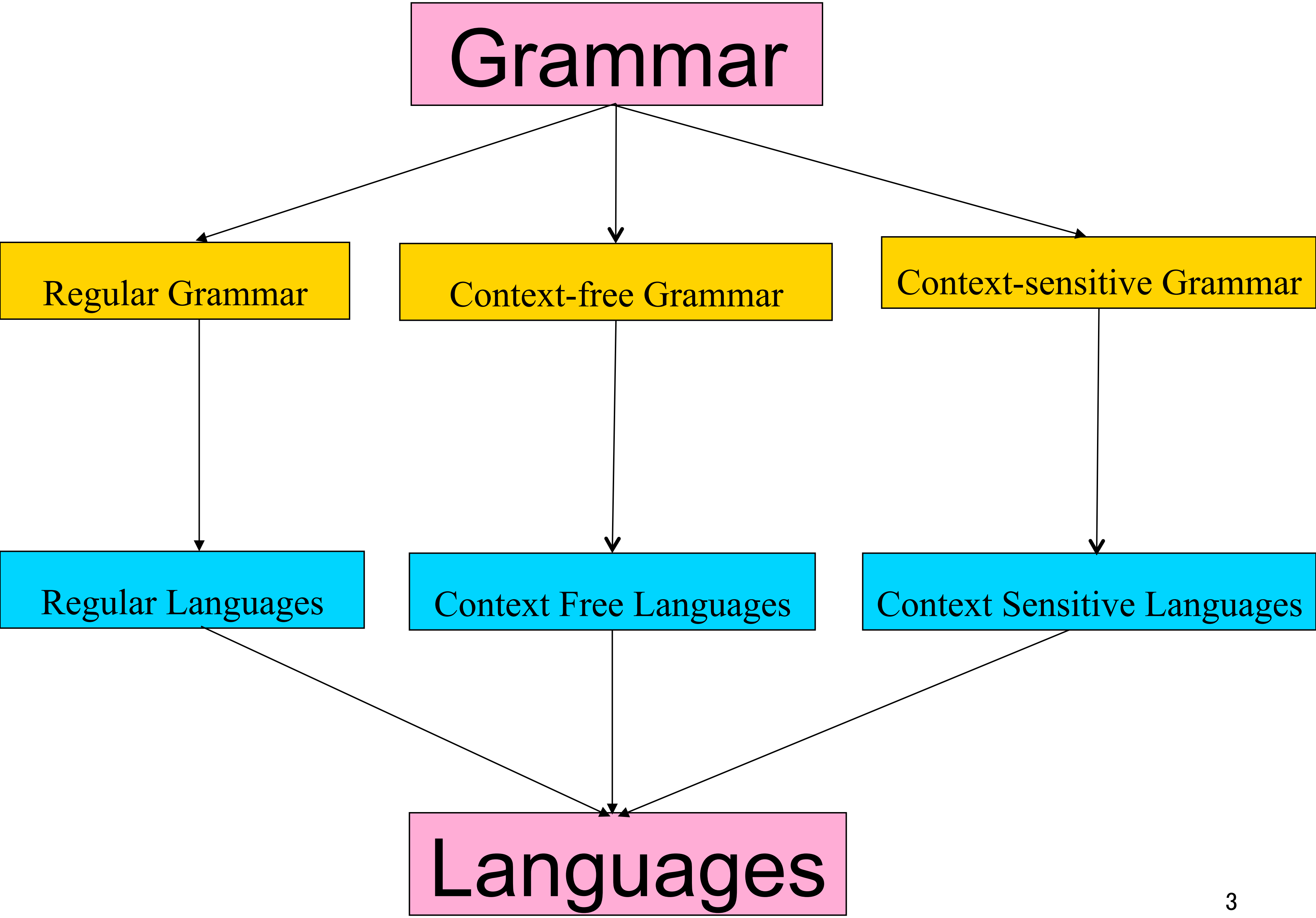

Grammar
Regular Grammar
Context-free Grammar
Context-sensitive Grammar
Regular Languages
Context Free Languages
Context Sensitive Languages
Languages

Context Free Languages (CFL)

Definition

L is a **Context Free Language** if and only if there is a context free grammar G=(V,T,S,P) such that:

$$L = L(G) = \{ w \in T^* : S \Rightarrow^* w \}$$

Context Free Languages (CFL)

Why Context Free Languages

Context-free languages allow us to describe languages that are nonregular like $\{ 0^n1^n : n \geq 0\}$.

CFLs are complex enough to give us a model for natural languages (cf. Noam Chomsky) and programming languages. The theory of CFLs is very closely related to the problem of "parsing" a computer program.

Later we will see that CFLs are the languages that can be recognized by automata that have one single stack:

$\{ 0^n1^n : n \geq 0 \}$ is a CFL

$\{ 0^n1^n0^n : n \geq 0 \}$ is not a CFL

Context Free Languages (CFL)

Properties of CFL:

If L_1 and L_2 are Context Free Languages then:

1. The language $L_1 \cup L_2$ is context free
2. The language $L_1 . L_2$ is context free
3. The language L_1^* and L_2^* are context free
4. The language $L_1 \cap L_2$ may be NOT a context free
5. The languages $\bar{L}_1$ and $\bar{L}_2$ may be NOT a context free

Context Free Languages (CFL)

Exercise

Consider the Context Free Languages:

$L_1 = \{a^n b^n c^m : n \geq 0, m \geq 0\}$

$L_2 = \{a^n b^m c^m : n \geq 0, m \geq 0\}$

1. Show that the languages $L_1 \cup L_2$, $L_1 . L_2$ and L_1^* are context free?

2. Show that the languages $L_1 \cap L_2$ and $\bar{L}_1$ are NOT context free

Pumping Lemma for CFL

Let L be a context-free language

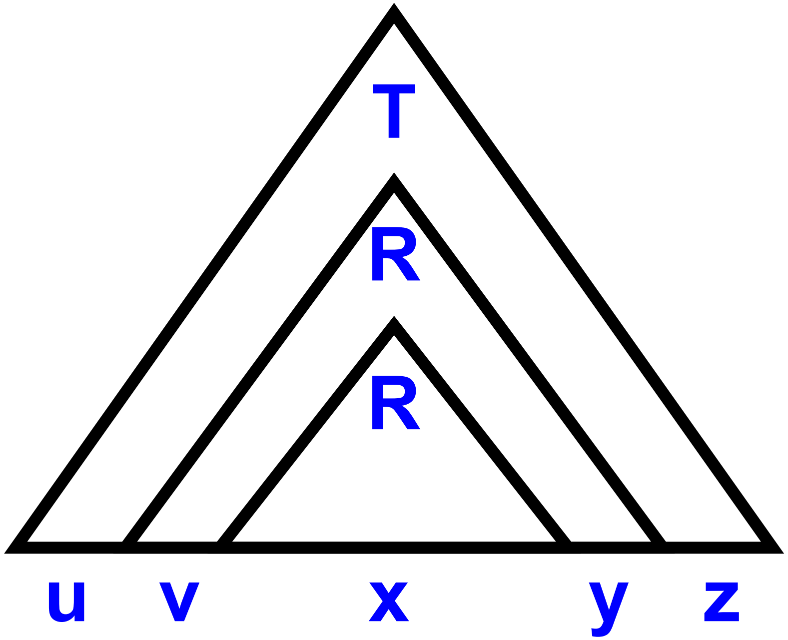
Then there exists P such that
if $w \in L$ and $|w| \geq P$

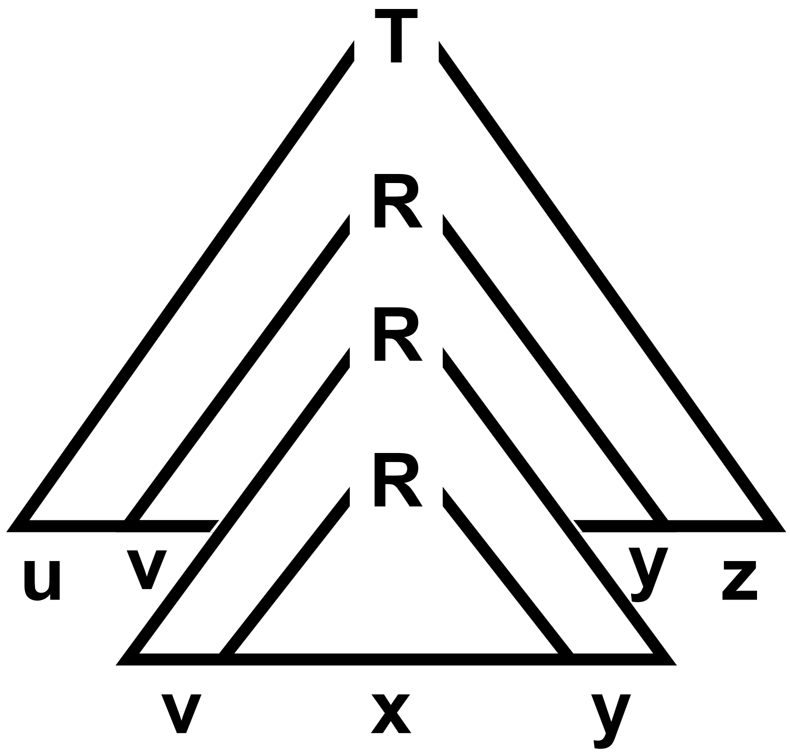
then $w = uvxyz$, where:

1. $|vy| > 0$
2. $|vxy| \leq P$
3. $uv^ixy^iz \in L$ for any $i \geq 0$

Pumping Lemma for CFL

Idea: If w is long enough, then any parse tree for w must have a path that contains a variable more than once





Pumping Lemma for CFL

Formal Proof: Let b be the maximum number of symbols on the right-hand side of a rule

If the height of a parse tree is h, the length of the string generated is at most: b^h

Let |V| be the number of variables in G

Define $P = b^{|v|+2}$

Let w be a string of length at least P

Let T be the parse tree for w with the smallest number of nodes.

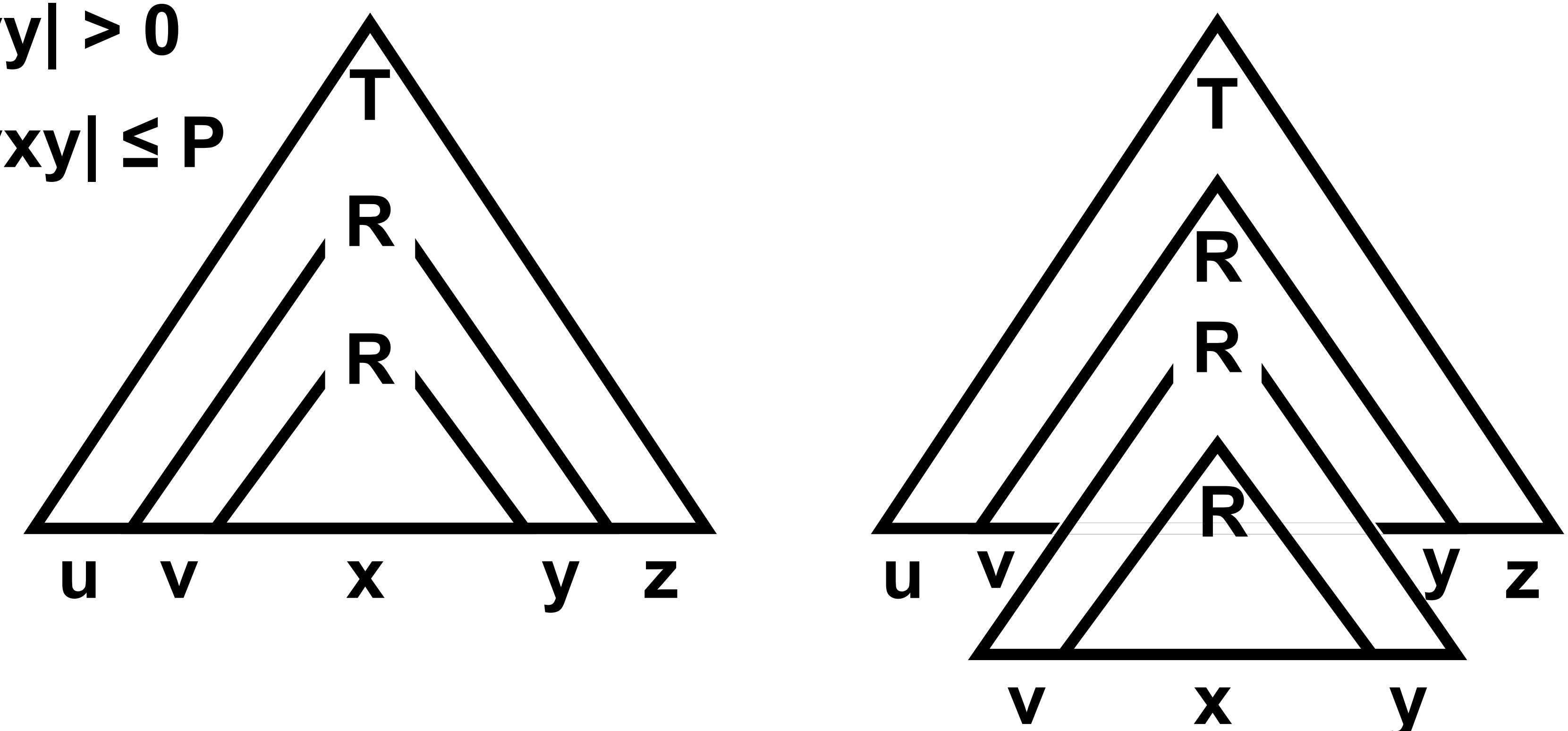
T must have height at least |V|+2

Pumping Lemma for CFL

The longest path in T must have ≥ |V|+1 variables

Select R to be the variable that repeats among the lowest |V|+1 variables

1. **|vy| > 0**
2. **|vxy| ≤ P**



Let T be the parse tree for w with the smallest number of nodes. T must have height at least |V|+2

Pushdown Automata

Push Down Automata (PDA)

Pushdown automata are for context-free languages while finite automata are for regular languages.

Big difference though: PDAs have to be nondeterministic (deterministic PDAs are not powerful enough).

PDAs are automata that have a single *stack* as memory.

Push Down Automata (PDA)

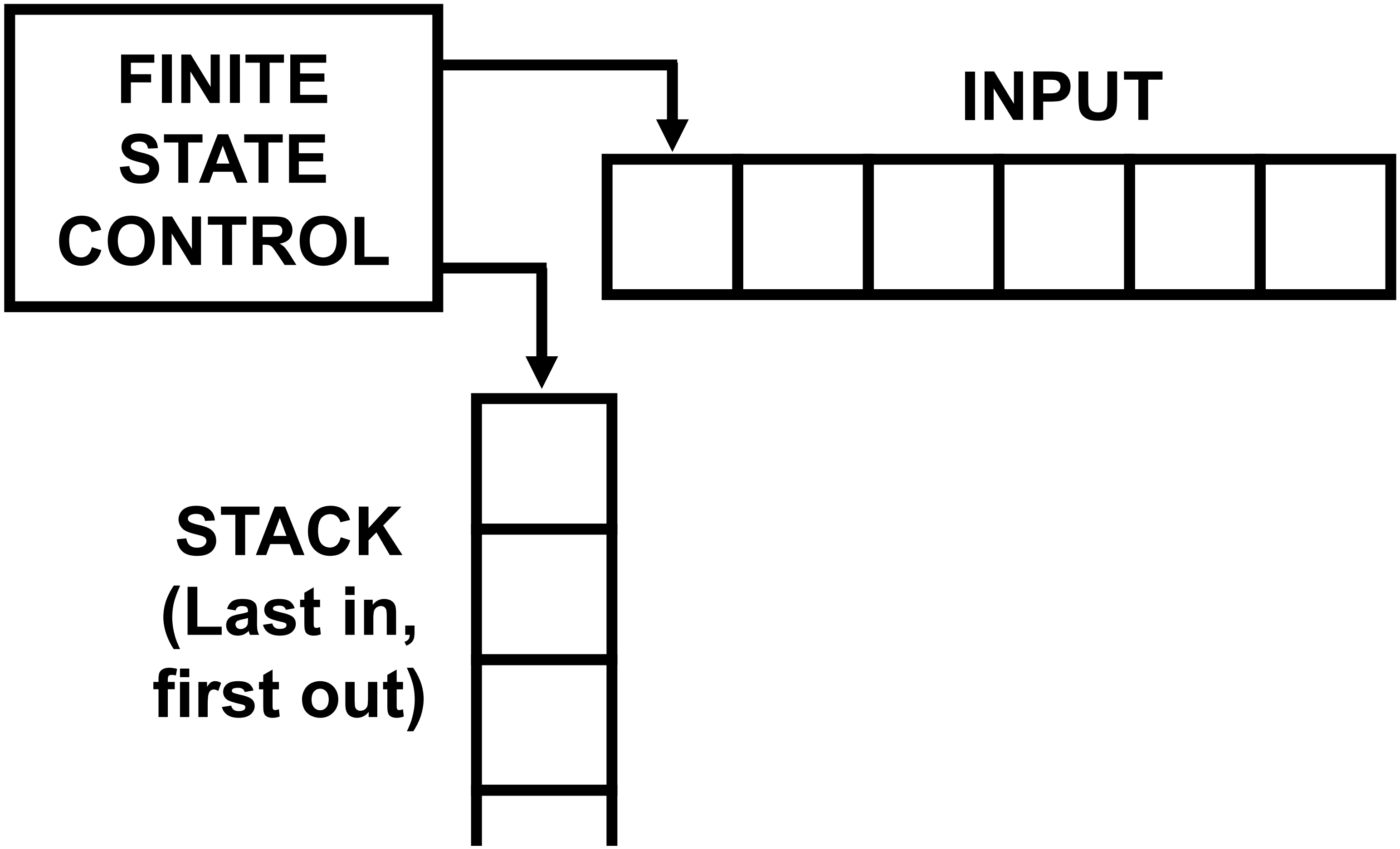
Definition

A **Nondeterministic Pushdown Automaton, Acceptor** (**NPDA**) M is defined by a tuple $(Q,\Sigma,\Gamma,\delta,q_0,z,F)$:

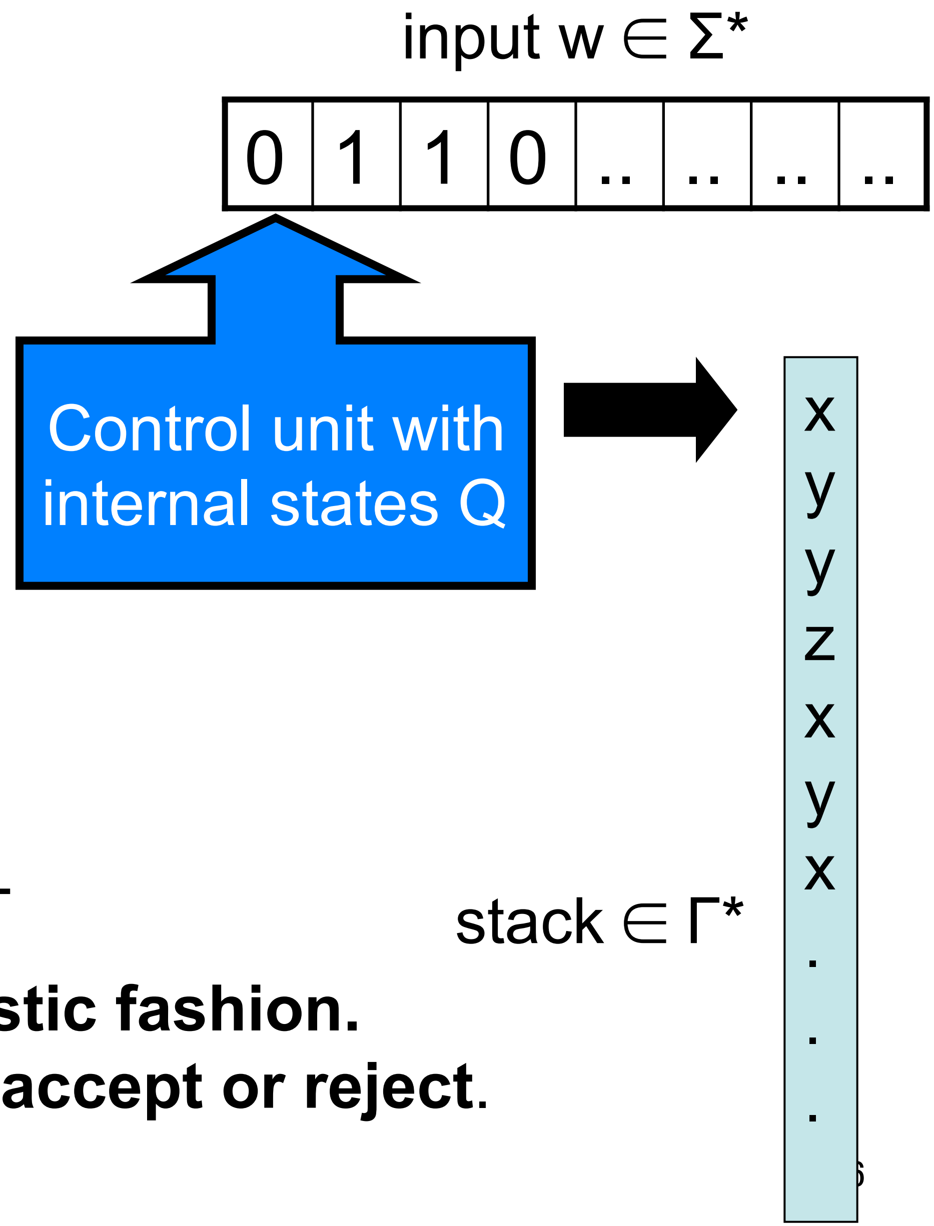
- Q is the finite set of internal states
- Σ is the finite input alphabet
- Γ is the finite **stack alphabet**
- $\delta:Q \times (\Sigma\cup\{\lambda\}) \times \Gamma \to \mathbf{P}(Q \times \Gamma^*)$ is the **transition function** of M, where each δ-value is a finite set
- $q_0\in Q$ is the starting state of M
- $z\in\Gamma$ is the **stack start symbol**
- $F\subseteq Q$ are the accepting, final states of M

It is the transition function δ that we need to understand…

Push Down Automata (PDA)



Push Down Automata (PDA)



The PDA M reads the input $w \in \Sigma^*$ from left to right.
Depending on

- input $w_j \in \Sigma \cup \{\lambda\}$,
- stack symbol $s_j \in \Gamma$,
- and state $q_k \in Q$

…the PDA M

- jumps to a new state $\in Q$,
- consume input w_i
- removes s_j from the stack
- and pushes new elements $\in \Gamma$

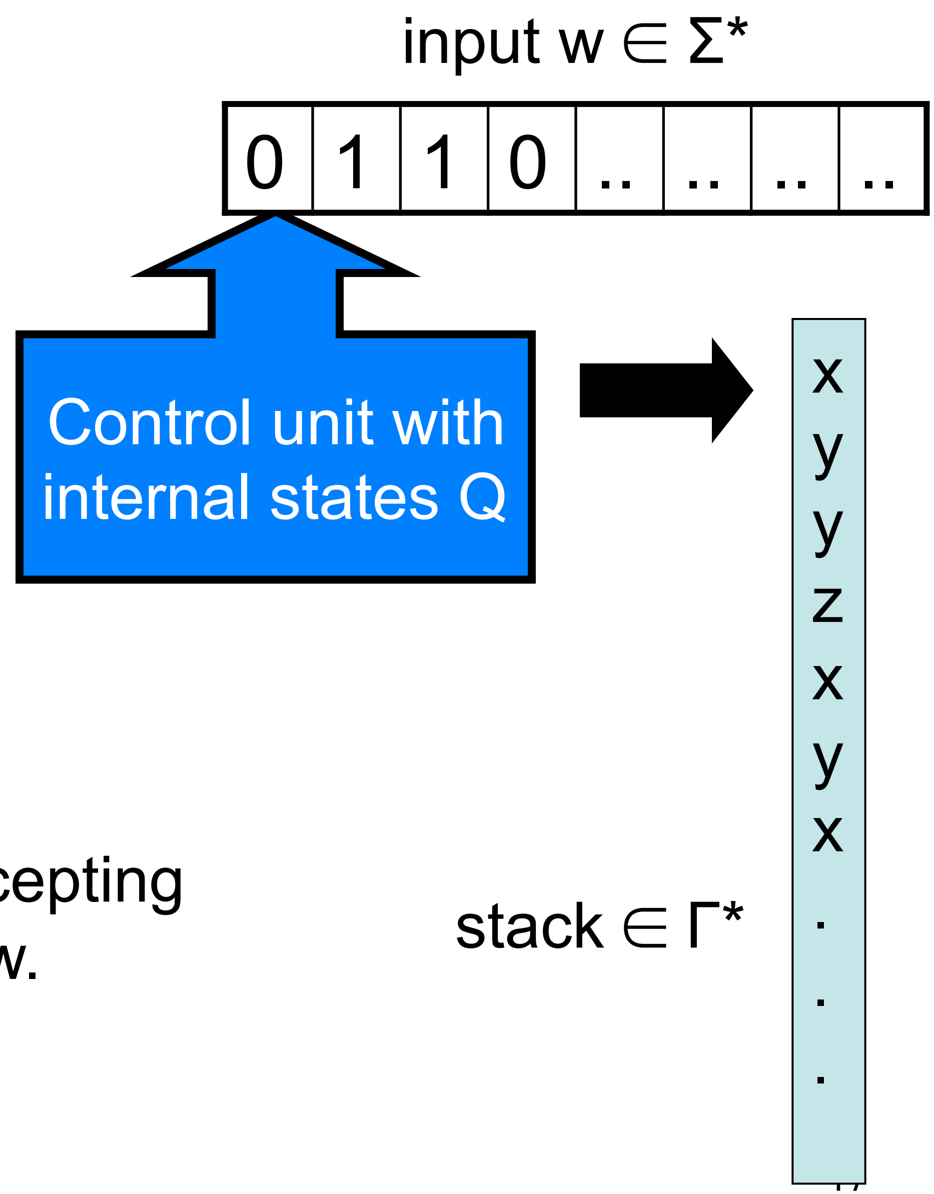
This is done in nondeterministic fashion.
After reading w, the PDA will accept or reject.

Push Down Automata (PDA)

For given $w_j \in \Sigma \cup \{\lambda\}$, $s_j \in \Gamma$ and $q_k \in Q$, the nondeterminism allows several possibilities for M. (Note the possibility of non-deterministic λ-transitions.)

After the PDA has read input w it can be in different state $\subseteq Q$.

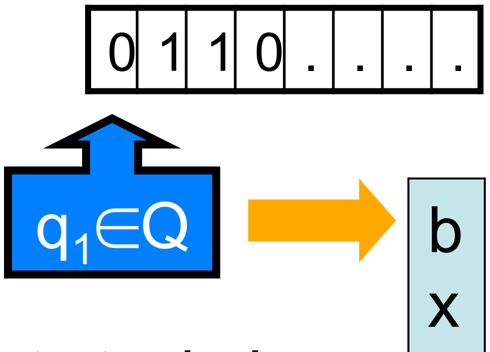
If it is possible to end in an accepting state $\in F \subseteq Q$, then M accepts w.



Push Down Automata (PDA)

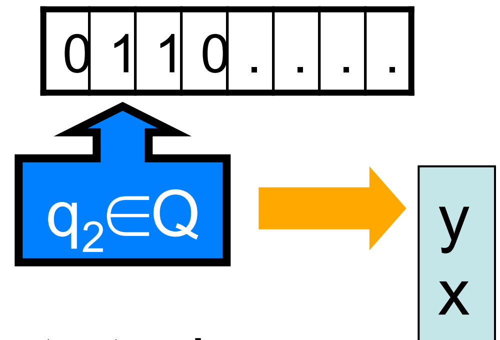
Transitions

If $(q_2,y) \in \delta(q_1,0,b)$ then the following transition is allowed:



content stack: bx,
with $b \in \Gamma$ and $x \in \Gamma^*$

goes to



content stack: yx,
with $y \in \Gamma^*$ and $x \in \Gamma^*$

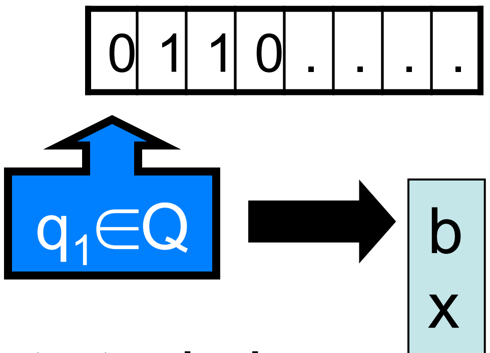
The states of the NPDA can be described by the triplets (q_1,aw,bx) and (q_2,w,yx) respectively and we denote this possible transition by $(q_1,aw,bx) \vdash (q_2,w,yx)$.

There can be several options according to δ, but it is required that the set $\delta(q_1,a,b)$ is finite.

Push Down Automata (PDA)

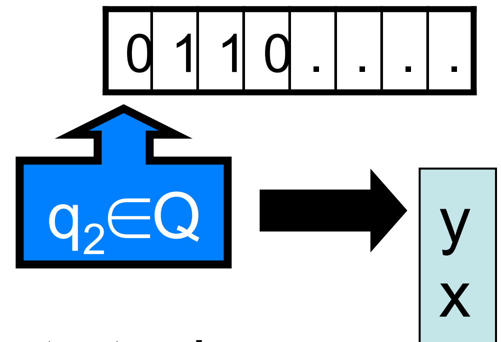
λ-Transitions

If $(q_2,y) \in \delta(q_1,\lambda,b)$ then the following transition is allowed:



content stack: bx,
with $b \in \Gamma$ and $x \in \Gamma^*$

goes to



content stack: yx,
with $y \in \Gamma^*$ and $x \in \Gamma^*$

Here the NPDA does not read a input letter and makes a λ-transition (compare λ-transitions for NFA).

In general, if the NPDA is allowed to make several steps we write: $(q,w_1\ldots w_n,x) \vdash^* (q',w_i\ldots w_n,x')$.

Push Down Automata (PDA)

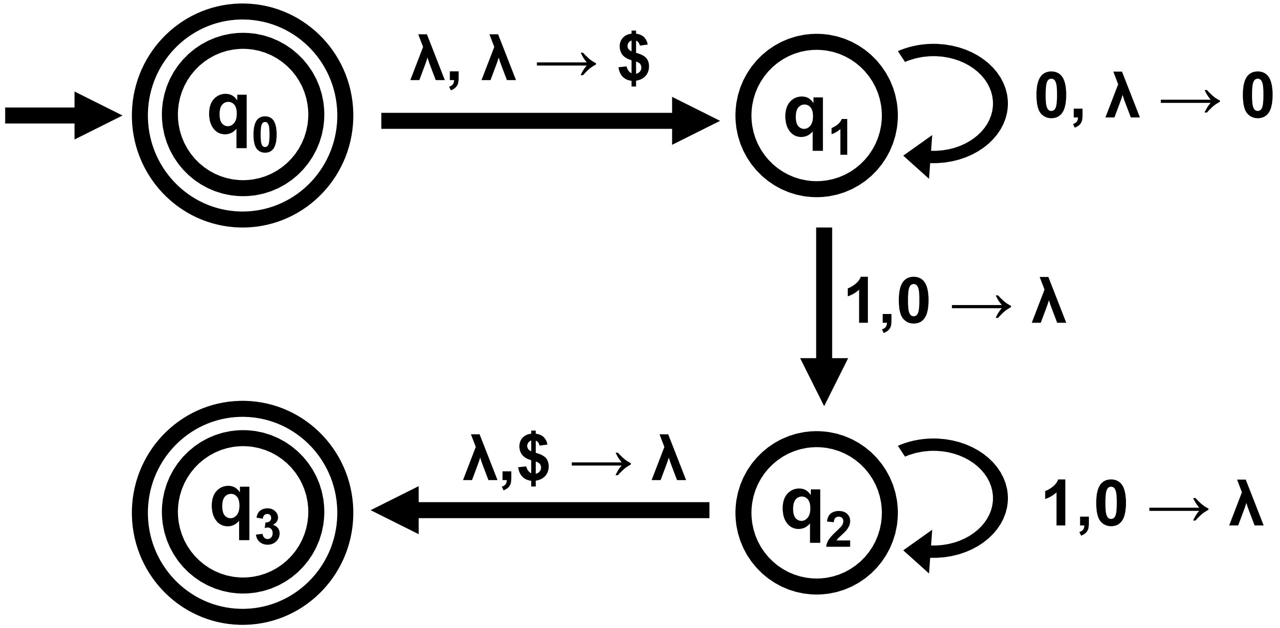
Accepted Language

- Definition: Given a NPDA $M = (Q,\Sigma,\Gamma,\delta,q_0,z,F)$, the language accepted by M is defined by:
  $L(M) = \{ w\in\Sigma^* : (q_0,w,z) \vdash^* (p,\lambda,u) \text{ with } p\in F,\ u\in\Gamma^* \}$.

- Note that the input part has to be empty (λ) in the end.
- The content of the stack does not matter.
- We only require that there is a possible transition.
- The only role of $z\in\Gamma$ is to start with a nonempty stack.

PDA: Example 1

$\delta:(q_0,\lambda,\lambda)=\{(q_1,\$)\}$

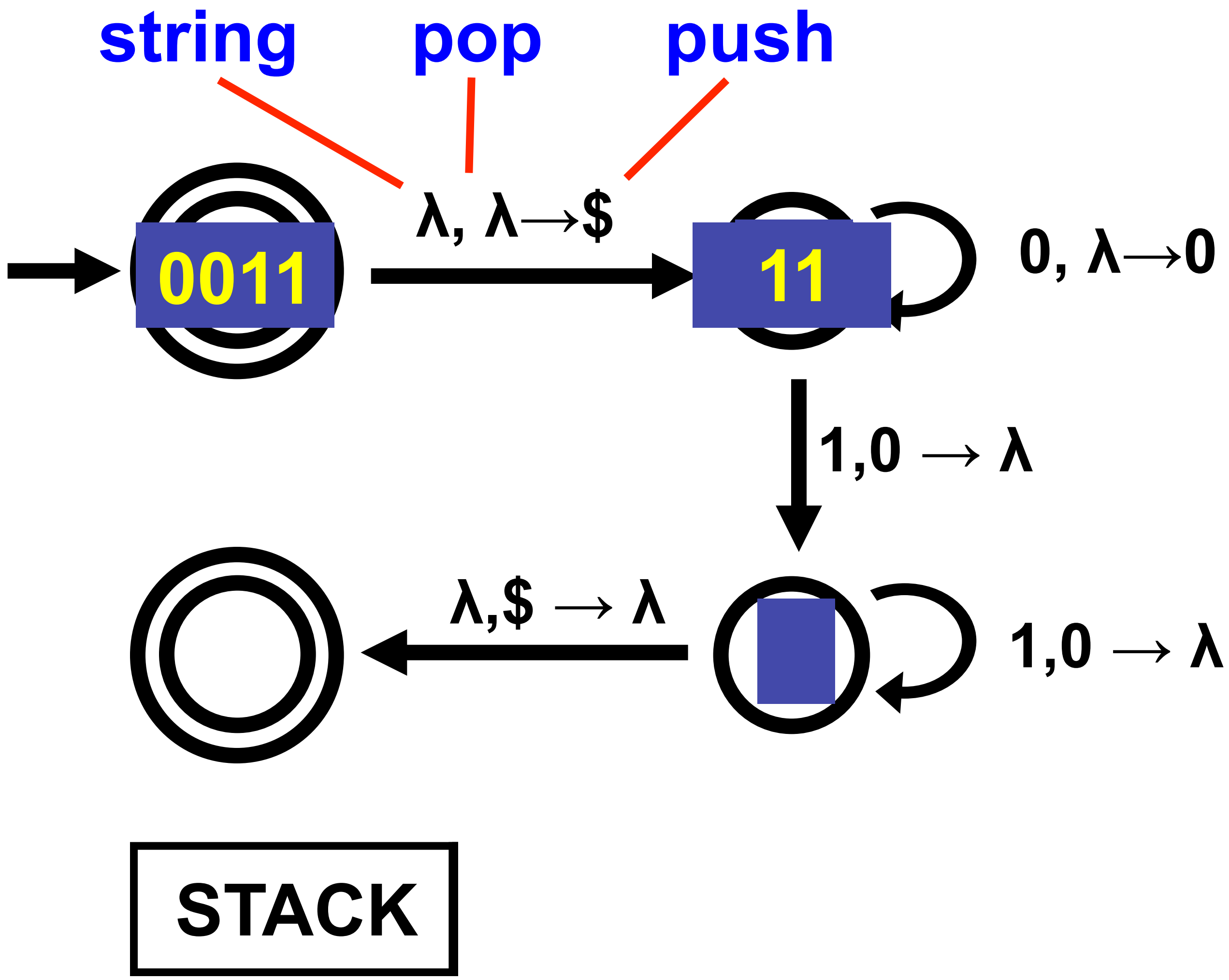


$Q = \{q_0, q_1, q_2, q_3\}$ $\Sigma = \{0,1\}$ $\Gamma = \{\$,0,1\}$

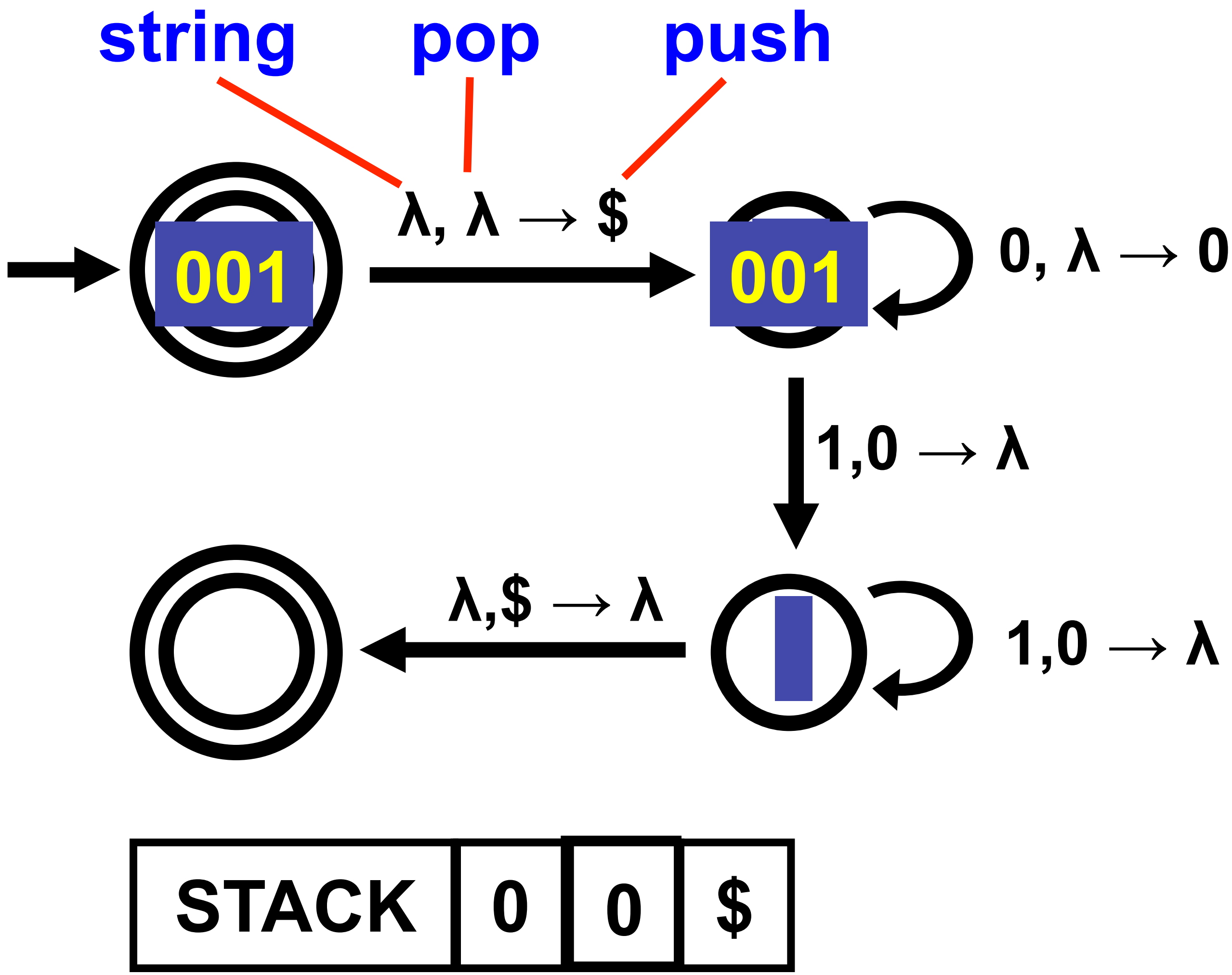
$\delta: Q \times (\Sigma \cup \{\lambda\}) \times \Gamma \to \mathbf{P}(Q \times \Gamma^*)$

$\delta(q_1,1,0) = \{ (q_2, \lambda) \}$ $\delta(q_2,1,1) = \varnothing$

PDA: Example 1



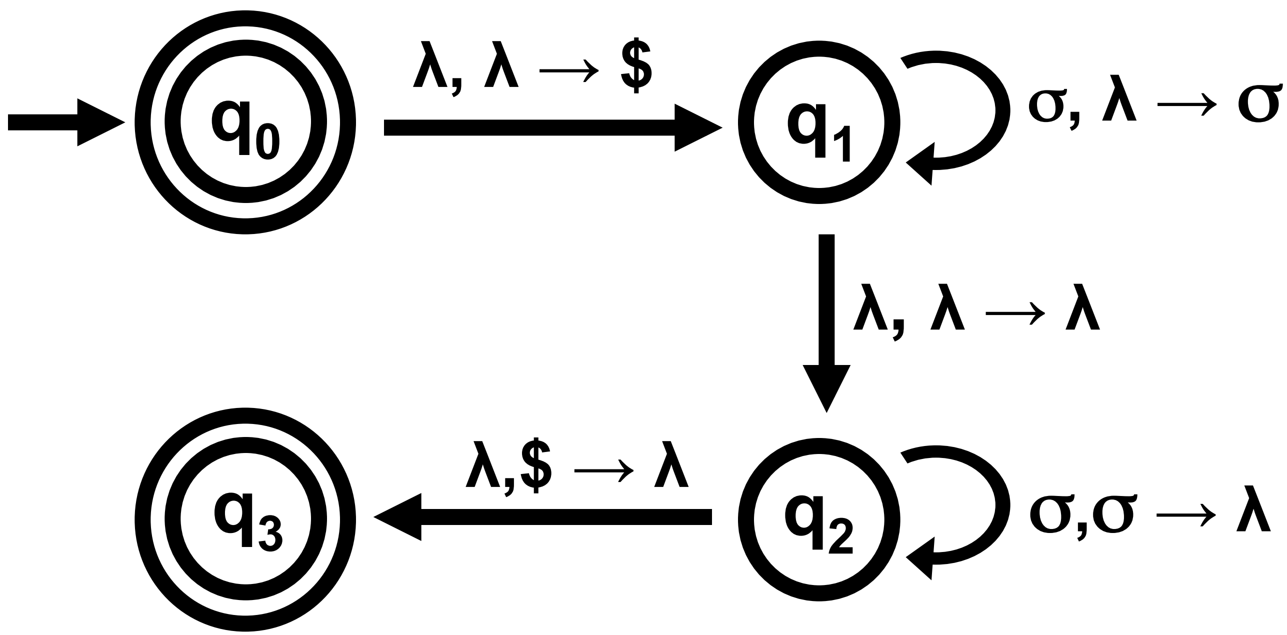
PDA: Example 1



PDA to recognize L = { 0^n1^n | n ≥ 0 }

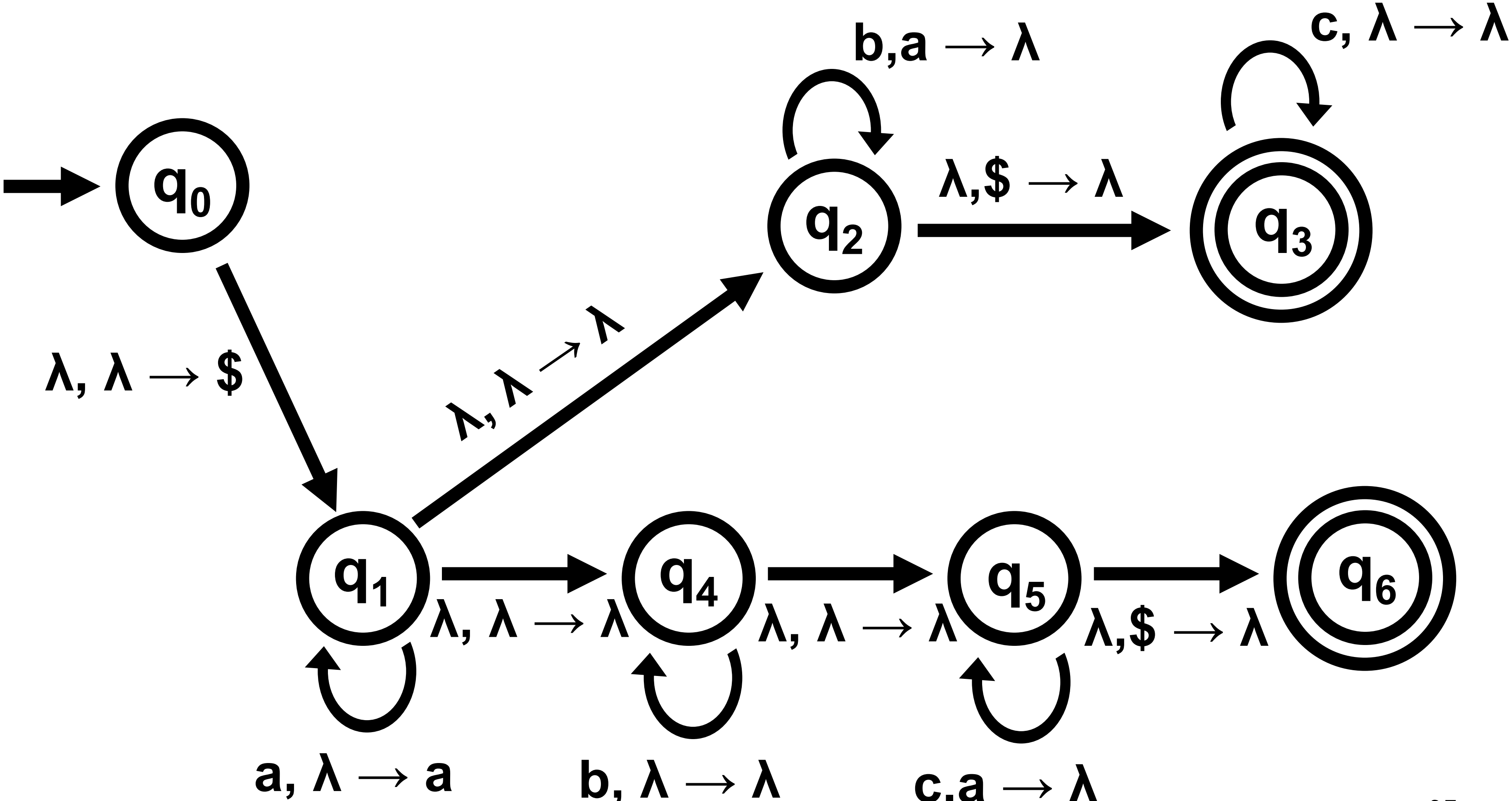
EVEN-LENGTH **PALINDROMES**

$\Sigma = \{a, b, c, \ldots, z\}$



PDA: Example 3

Build a PDA to recognize $L = \{ a^i b^j c^k \mid i, j, k \geq 0$ and $(i = j \text{ or } i = k) \}$



Push Down Automata (PDA)

Example 4

- How to make a NPDA that accepts $\{w \in \{a,b\}^* : \#a = \#b\}$?
- Central idea push/pull counting symbols 0 (for a) and 1 (for b) on the stack as you read the string w.
- Answer: NPDA $M = (\{q_0,q_f\},\{a,b\},\{0,1,z\},\delta,q_0,z,\{q_f\})$ with:
- $\delta(q_0,\lambda,z) = \{(q_f,z)\}$

$\delta(q_0,a,z) = \{(q_0,0z)\}$
$\delta(q_0,b,z) = \{(q_0,1z)\}$
$\delta(q_0,a,0) = \{(q_0,00)\}$
$\delta(q_0,b,0) = \{(q_0,\lambda)\}$
$\delta(q_0,a,1) = \{(q_0,\lambda)\}$
$\delta(q_0,b,1) = \{(q_0,b,11)\}$

Processing baab gives:

$$
\begin{aligned}
(q_0,baab,z) &\vdash (q_0,aab,1z) \\
&\vdash (q_0,ab,z) \\
&\vdash (q_0,b,0z) \\
&\vdash (q_0,\lambda,z) \\
&\vdash (q_f,\lambda,z).
\end{aligned}
$$

Push Down Automata (PDA)

Exercise

- How to recognize $ww^R \in \{a,b\}^*$?
- Idea: Read w and put it on the stack (first in, last out).
At the half-way point, start checking the remaining
input string w^R against the stack content w^R.

- Crucial observation: We have to guess the half-way point.

PDA $\Longleftrightarrow$ CFL

A language L is context-free if and only if there is a non-deterministic pushdown automaton M that recognizes L.

Two step proof:

Theorem 1: Given a CFG G, we can construct a NPDA M_G such that $L(G)=L(M_G)$.

Theorem 2: Given a PDA M, we can construct a CFG G such that $L(G)=L(M_G)$.

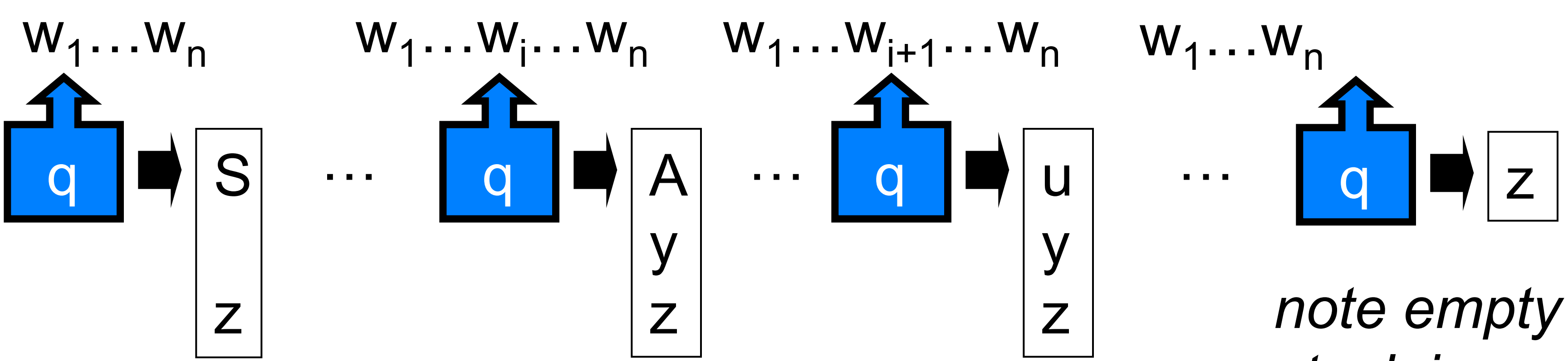
Central idea: Use Greibach normal form for the CFG and put the variables on the stack while reading letters.

PDA ⟸ CFL

Proof Idea

- Without loss of generality, we assume that the CFG is in Greibach normal form, for which we know that the derivation of $w_1 \ldots w_n$ has to look like (assuming $A \to w_i u$):
- $S \Rightarrow \ldots \Rightarrow w_1 \ldots w_{i-1} A y \Rightarrow w_1 \ldots w_i u y \Rightarrow \ldots \Rightarrow w_1 \ldots w_n$ with $y, u \in V^*$.

- Idea for PDA implementation: when reading the input string $w_1 \ldots w_n$ from left to right on put possible $u \in V^*$ on stack:



note empty stack in end

$\boxed{(q, w_i, A) \vdash (q, u)}$

PDA $\Longleftarrow$ CFL

Proof details

- Given a CFG $G=(V,T,S,P)$ in Greibach normal form, the NPDA $M=(\{q_0,q_1,q_f\},T,V\cup\{z\},\delta,q_0,z,\{q_f\})$ accepts $L(G)$:

- $\boldsymbol{\delta(q_0,\lambda,z) = \{(q_1,Sz)\}}$ (start with S on the stack)
- $(q_1,u) \in \delta(q_1,a,A)$ for all $A\rightarrow au$ rules
- $\boldsymbol{\delta(q_1,\lambda,z) = \{(q_f,z)\}}$

PDA $\Longleftarrow$ CFL

Example

$S \to \lambda \mid a \mid b \mid aSa \mid bSb$ $\xRightarrow{\text{GNF}}$ $S \to \lambda \mid a \mid b \mid aSA \mid bSB$

$A \to a, \; B \to b$

$\delta(q_0,\lambda,z) = \{(q_1,Sz)\}$

$(q_1, \lambda) \in \delta(q_1, \lambda, S)$

$(q_1, \lambda) \in \delta(q_1, a, S)$

$(q_1, \lambda) \in \delta(q_1, b, S)$

$(q_1, SA) \in \delta(q_1, a, S)$

$(q_1, SB) \in \delta(q_1, b, S)$

$(q_1, \lambda) \in \delta(q_1, a, A)$

$(q_1, \lambda) \in \delta(q_1, b, B)$

$\delta(q_1,\lambda,z) = \{(q_f,z)\}$

PDA $\Longrightarrow$ CFL

- To prove that for every NPDA there is a corresponding CFG we use the same ideas as for the previous proofs.

- The proof uses the assumption that the NPDA has only one accepting state q_f that is entered when the stack is empty, and all transitions are of the form (q,a,A) (q',λ) or (q',BC).
- This assumption can be made without loss of generality.

PDA $\Longrightarrow$ CFL

Given PDA P = (Q, Σ, Γ, δ, q, F)

Construct a CFG G = (V, Σ, R, S) such that **L(G)=L(P)**

First, **simplify P** to have the following form:

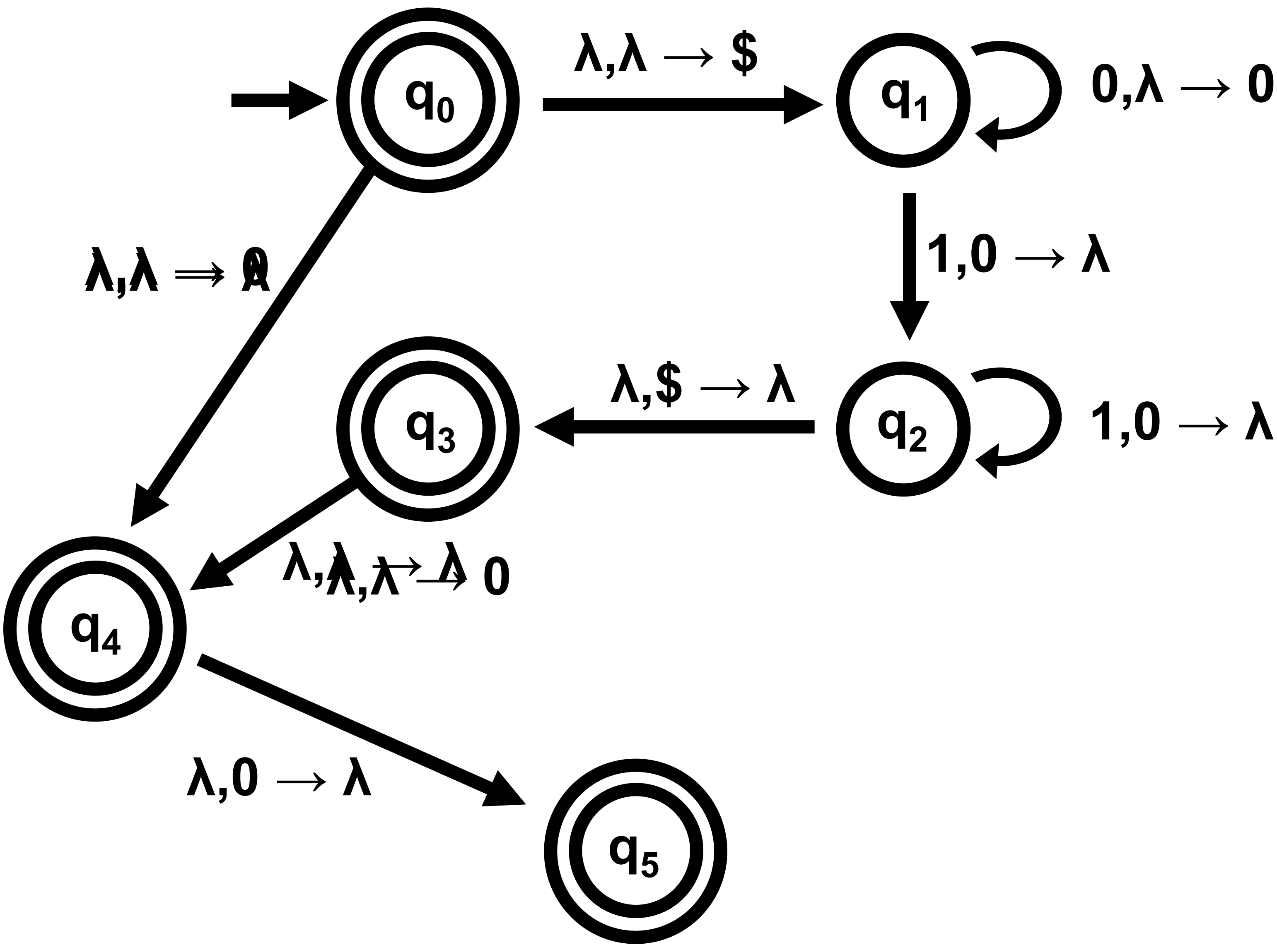
(1) It has a single accept state, q_{accept}

(2) It empties the stack before accepting

(3) Each transition either pushes a symbol or pops a symbol, but not both at the same time

PDA $\Longrightarrow$ CFL

SIMPLIFY



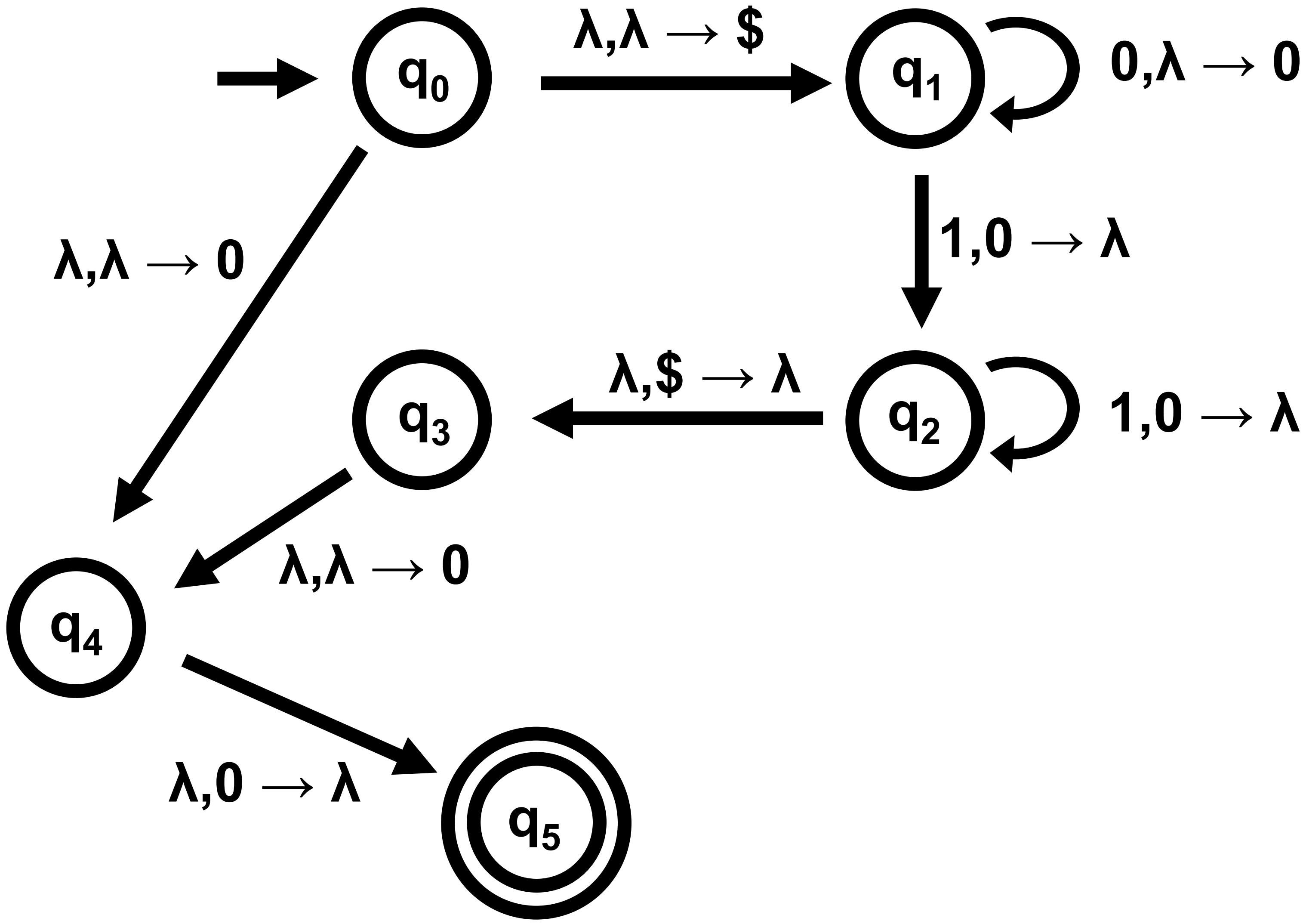
PDA $\Longrightarrow$ CFL

Idea: for each pair of states p and q in P, the grammar will have a variable A_{pq} that generates all strings that can take P from p with an empty stack to q with an empty stack

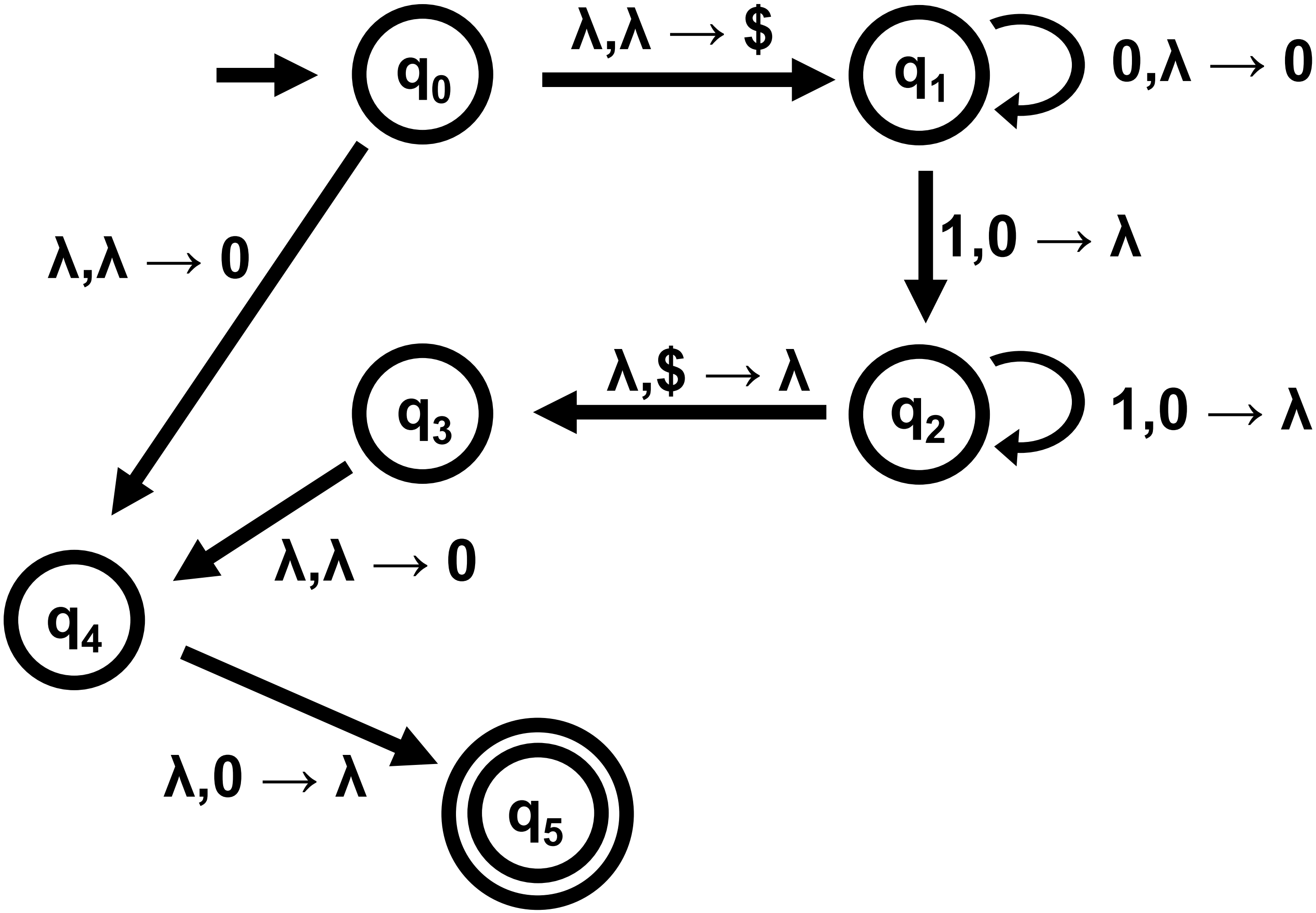
$$V = \{A_{pq} \mid p,q \in Q\}$$

$$S = A_{q_0 q_{accpet}}$$

PDA $\Longrightarrow$ CFL



PDA $\Longrightarrow$ CFL



What strings does $A_{q_0q_1}$ generate? none

What strings does $A_{q_1q_2}$ generate? $\{0^n1^n \mid n > 0\}$

What strings does $A_{q_1q_3}$ generate? none

PDA $\Longrightarrow$ CFL

A_{pq} generates all strings that take p with an empty stack to q with an empty stack

Let x be such a string

- P's first move on x must be a **push**
- P's last move on x must be a **pop**

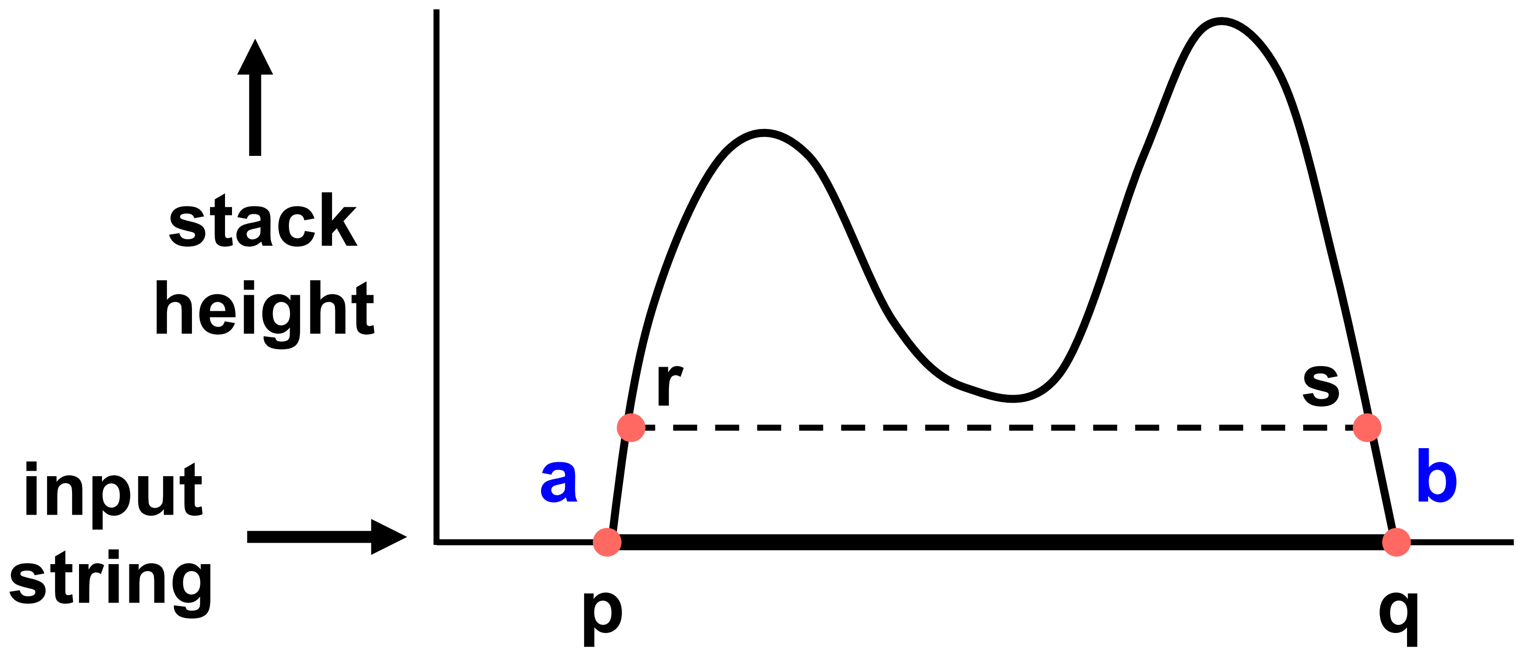
Two possibilities:

1. The symbol popped at the end is the one pushed at the beginning

2. The symbol popped at the end is **not** the one pushed at the beginning

PDA $\Longrightarrow$ CFL

1. The symbol popped at the end is the one pushed at the beginning



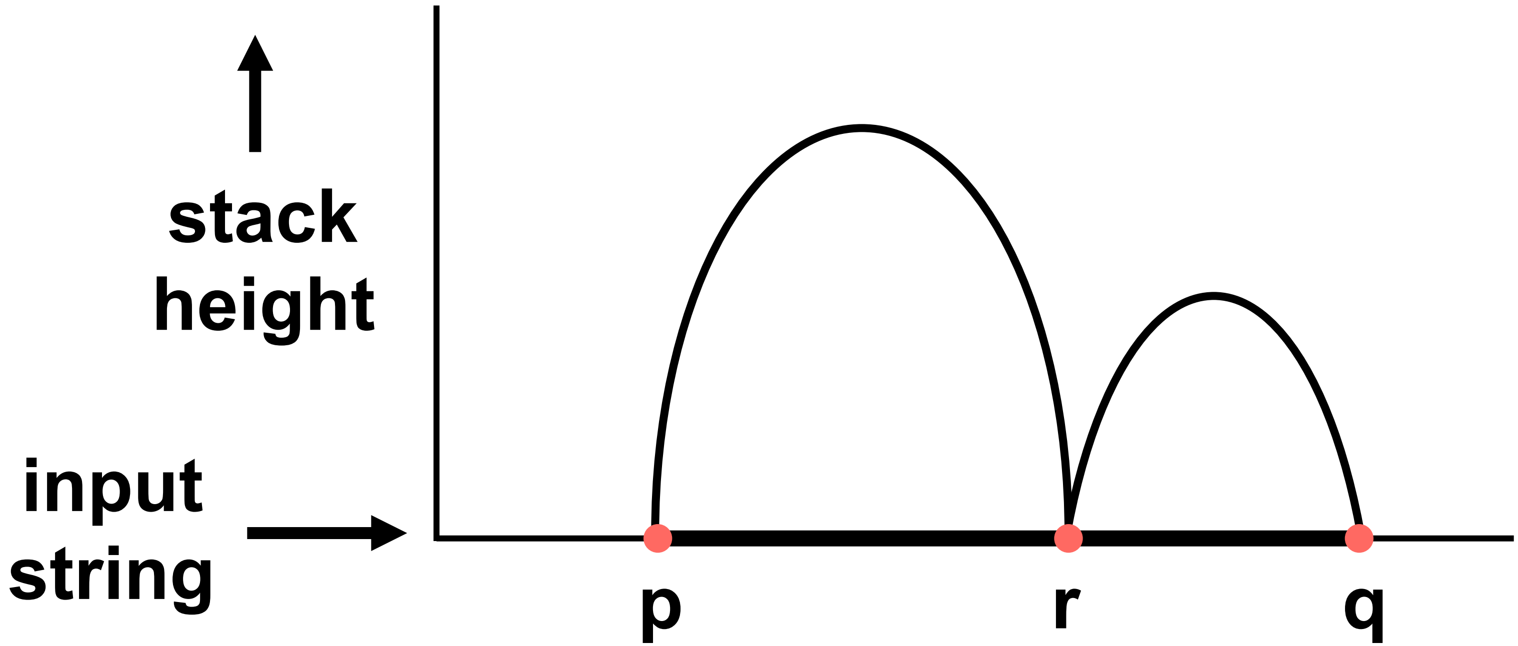
$\delta(p,a,z) = \{(r,H)\}$

$A_{pq} \rightarrow aA_{rs}b$

$\delta(s,b,H) = \{(q,z)\}$

PDA ⟹ CFL

2. The symbol popped at the end is not the one pushed at the beginning



$\delta(p,a,z) = \{(n,\boxed{H})\}$

$\delta(m,b,\boxed{E}) = \{(q,z)\}$

$$A_{pq} \rightarrow A_{pr}A_{rq}$$

PDA $\Longrightarrow$ CFL

Formally: $V = \{A_{pq} \mid p,q \in Q\}$

$S = A_{q_0 q_{accpet}}$

For each $p,q,r,s \in Q$, $t \in \Gamma$ and $a,b \in \Sigma_\lambda$

If $(r,t) \in \delta(p,a,\lambda)$ and $(q, \lambda) \in \delta(s,b,t)$

Then add the rule $A_{pq} \to aA_{rs}b$

For each $p,q,r \in Q$,

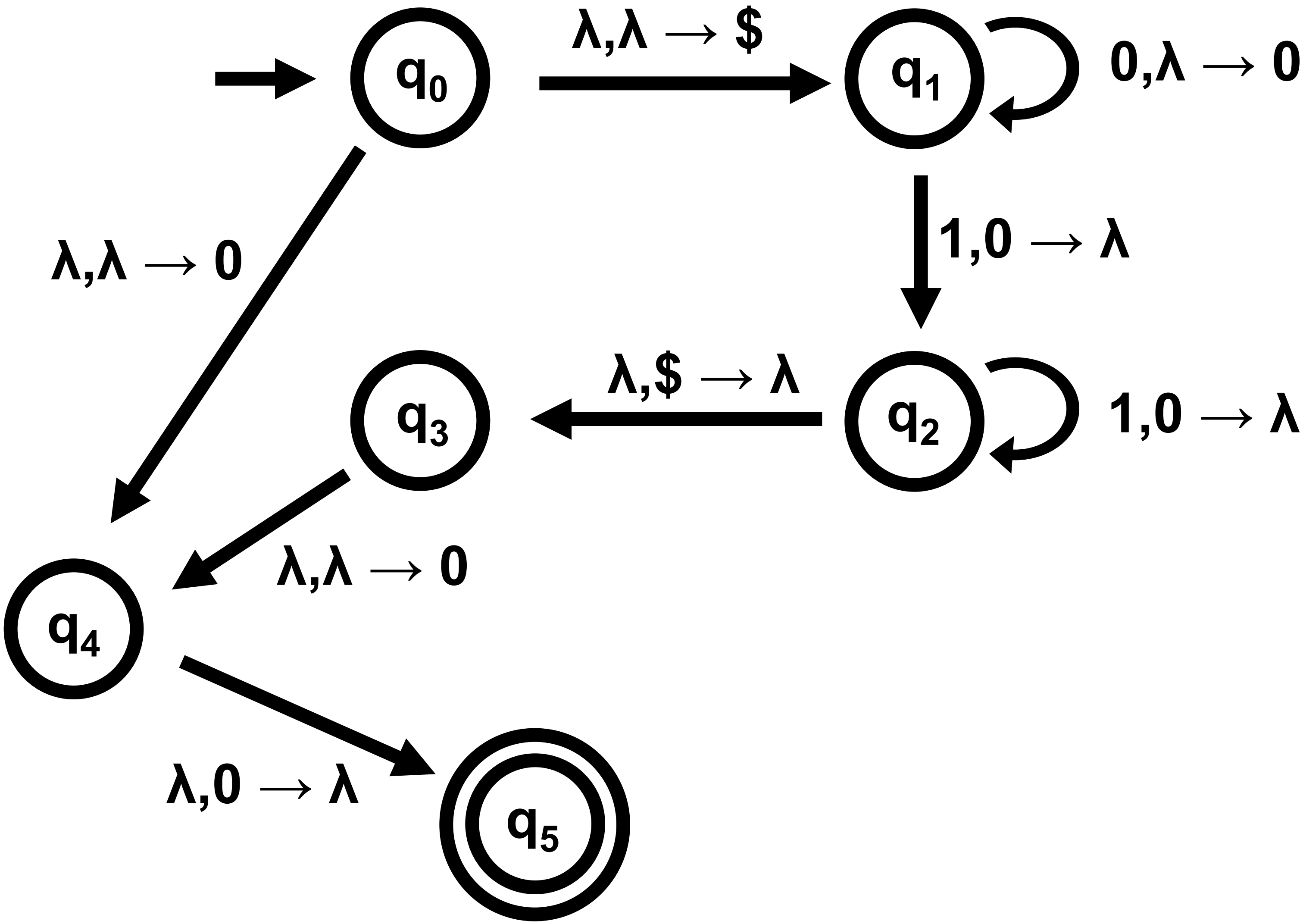
add the rule $A_{pq} \to A_{pr}A_{rq}$

For each $p \in Q$,

add the rule $A_{pp} \to \lambda$

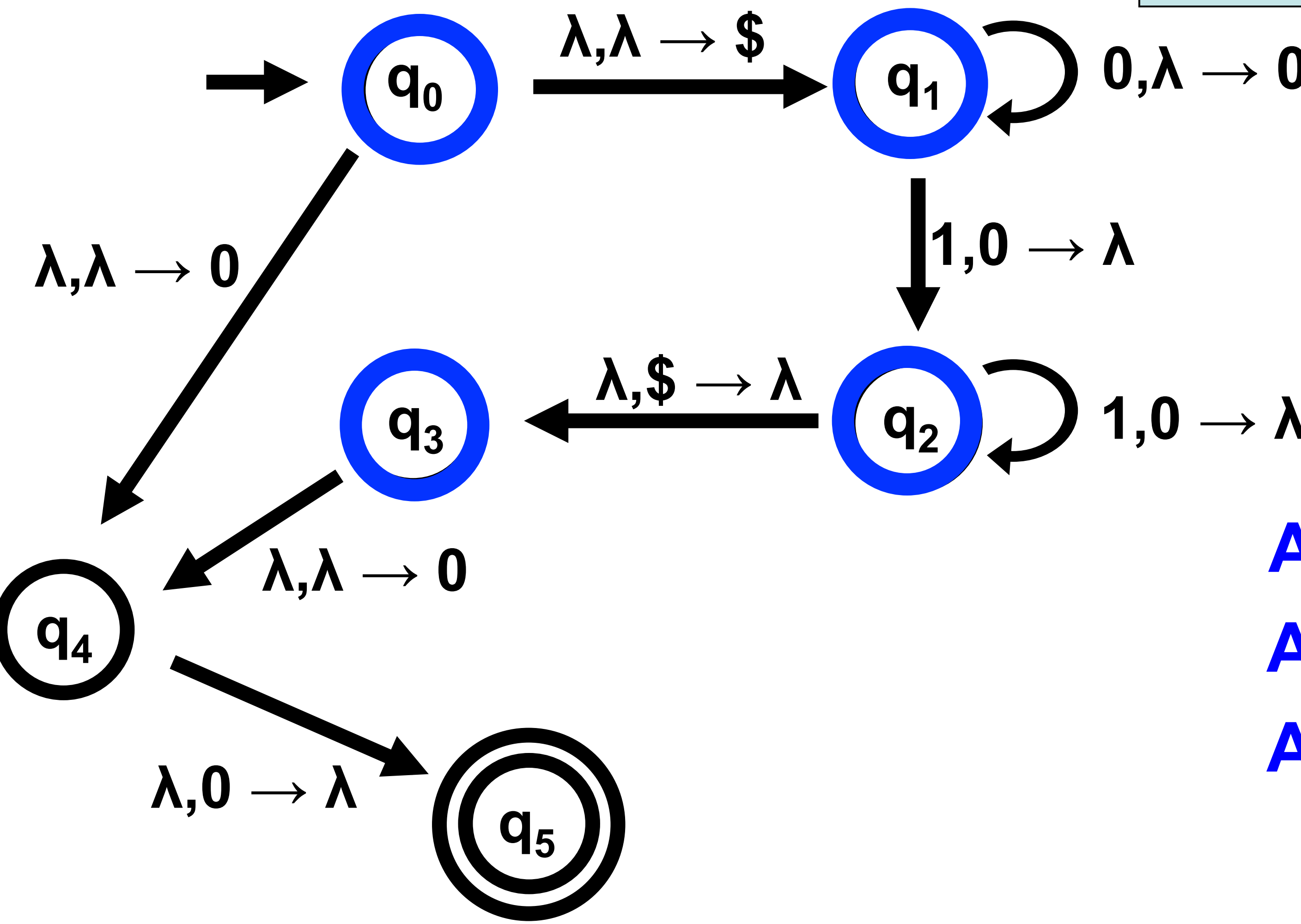
PDA $\Longrightarrow$ CFL

Example



PDA ⟹ CFL

Example



$A_{qq} \rightarrow \lambda$

$A_{pq} \rightarrow A_{pr}A_{rq}$

$A_{q_0q_3} \rightarrow \lambda A_{q_1q_2}\lambda$

$A_{q_1q_2} \rightarrow 0A_{q_1q_2}1$

$A_{q_1q_2} \rightarrow 0A_{q_1q_1}1$

What strings does $A_{q_0q_1}$ generate? none

What strings does $A_{q_1q_2}$ generate? $\{0^n1^n \mid n > 0\}$

What strings does $A_{q_1q_3}$ generate? none