Automata and Languages

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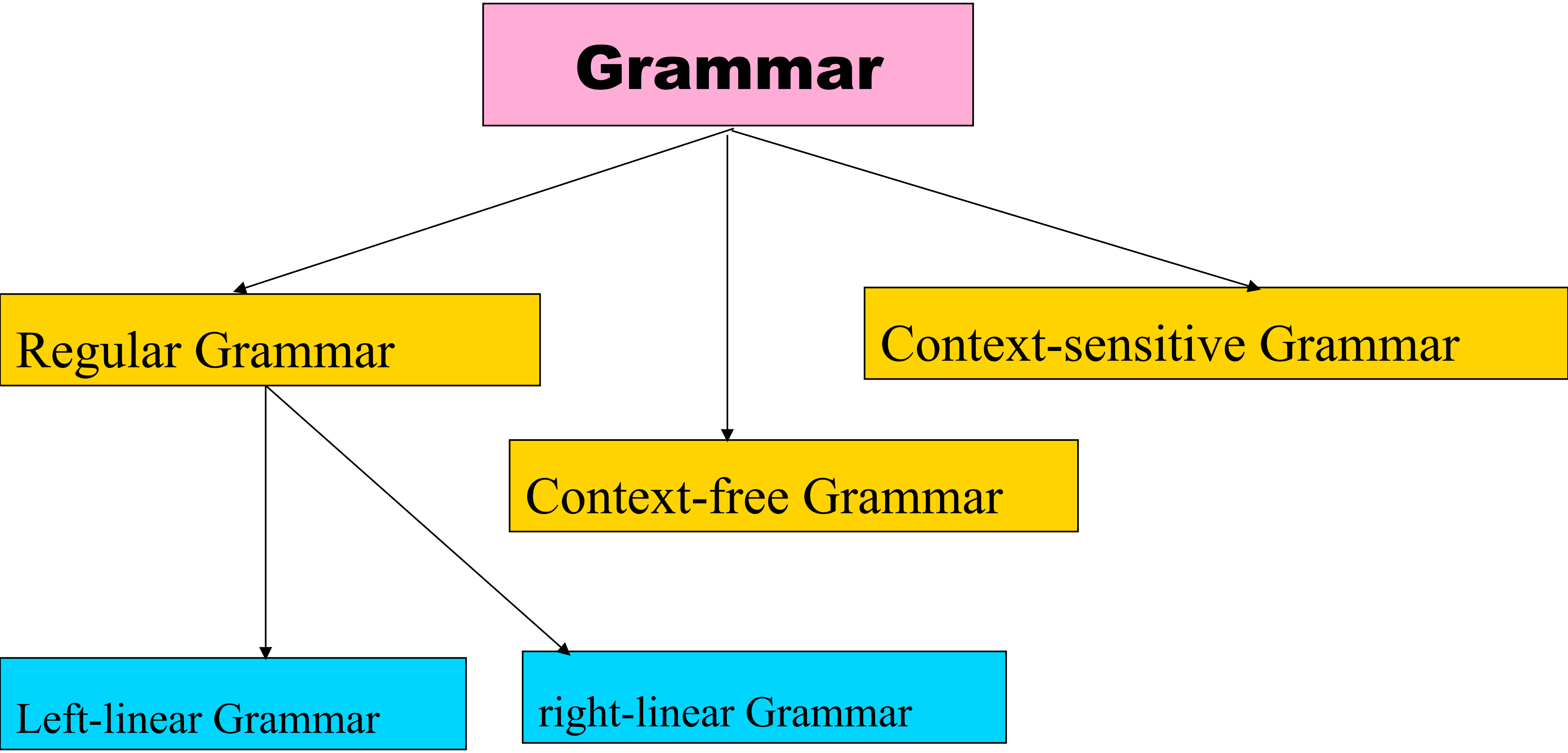
Grammar


Grammar
Regular Grammar
Context-sensitive Grammar
Context-free Grammar
Left-linear Grammar
right-linear Grammar

Today's Topics

- Grammars
- Right-Linear Grammars
- Left-Linear Grammars
- Regular Grammars
- Context-free Grammars
- Derivation: Leftmost & Rightmost
- Derivation Tree

Grammars

Definition

A grammar G is defined as G = (V, T, P, S) where:

- V : Finite set of variables/non-terminals
  (We use capital letters A,B,C,… for variables)
- T : Alphabet/Finite set of terminals
  (We use small letters a,b,c,… for terminals)
- P : Finite set of rules/productions
- S : Start symbol

$$S \in V$$

$$V \cap T = \phi$$

$$\text{Rule}: \quad \alpha \rightarrow \beta$$

$$\alpha \in (V \cup T)^{+}, \quad \beta \in (V \cup T)^{*}$$

Each grammar G defines a language L(G), which is the set of strings in T* (=Σ*) that G can generate from S.
It is all about the production rules.

Grammars

Definition

Given a grammar G = (V, T, P, S)

For a string $w=uxv$ we can apply the production rule $x \rightarrow y$ to w so we get a string $z=uyv$.

In this case we write $w \Longrightarrow z$, which reads *w* drives *z.*

If $w_1 \Longrightarrow w_2 \Longrightarrow \ldots \Longrightarrow w_n$,

we say that w_1 drives w_n and we write $w_1 \overset{*}{\Longrightarrow} w_n$

Grammars

Example

Given a grammar G = (V, T ,P,S)

V={A, B, C}

T={a, b, x}

S= A

And P is:

AaBx→aBAaBb

CaBx→aBAaCb

ABC→λ

Grammars

Definition

Let G=(V, T, P, S) be a grammar.

- $w \in (V \cup T)^*$ is a *sentential form*, if

  $S \Rightarrow_G^* w$

- $w \in T^*$ is a *sentence*, if $S \Rightarrow_G^* w$

- The *language of G*,

  L(G) = $\{w \in T^* \mid S \Rightarrow_G^* w\}$

Some Remarks

The language $L(G) = \{ w \in T^* : S \Rightarrow^* w \}$ contains only strings of terminals, not variables.

Notation: We summarize several rules for one variable:

$A \rightarrow B$

$A \rightarrow 01$ by $A \rightarrow B \mid 01 \mid AA$

$A \rightarrow AA$

Grammars

Example

Given the grammar:

$$G = (\{S\}, \{a, b\}, \{S \to \lambda, S \to aSb\}, S)$$

The language generated by this grammar is:

$$L(G) = \{a^n b^n \mid n \ge 0\}$$

Right-Linear Grammars

Definition

A Grammar G= **(V, T ,P,S)** is called ***right-linear grammar*** if every production is of the form $A \rightarrow xB$, or $A \rightarrow x$ where

A,B $\in$ V, x $\in$ **T***

Example: The grammar

$x \rightarrow 0x \mid 1y$

$y \rightarrow 0x \mid 1z$

$z \rightarrow 0x \mid 1z \mid \lambda$

Is a right-linear grammar.

Left-Linear Grammars

Definition

A Grammar G= **(V, T ,P,S)** is called ***left-linear grammar*** if every production is of the form $A \rightarrow Bx$, or $A \rightarrow x$ where

$A,B \in V$, $x \in \mathbf{T}^*$

Example: The grammar

$x \rightarrow x0 \mid y1$

$y \rightarrow x0 \mid z1$

$z \rightarrow x0 \mid z1 \mid \lambda$

Is a left-linear grammar.

Regular Grammars

Definition

A Grammar G= **(V, T ,P,S)** is called ***regular grammar*** if its is left- or right-linear

Example: The grammar
$x \rightarrow x0 \mid y1$
$y \rightarrow x0 \mid z1$
$z \rightarrow x0 \mid z1 \mid \lambda$
Is a left-linear grammar,
hence is Regular Grammar

Example: The grammar
$x \rightarrow 0x \mid 1y$
$y \rightarrow 0x \mid 1z$
$z \rightarrow 0x \mid 1z \mid \lambda$
Is a right-linear grammar,
Hence is Regular Grammar

Grammars

Example 1

Write a grammar that generate the language:

$$L = \{w \in \{a,b\}^* \mid \text{length}(w) \text{ is EVEN}\}$$

$$\begin{aligned} E \rightarrow \; & \lambda \\ & \mid aaE \mid abE \\ & \mid baE \mid bbE \end{aligned}$$

$$\begin{aligned} E &\rightarrow \lambda \mid aO \mid bO \\ O &\rightarrow aE \mid bE \end{aligned}$$

Example 2

Write a grammar that generate the language:

$$L = \{w \in \{a,b\}^* \mid w \text{ has EVEN number of } b\text{'s}\}$$

$$\begin{aligned} E &\rightarrow \lambda \mid aE \mid bO \\ O &\rightarrow aO \mid bE \end{aligned}$$

Grammars

Example 3

Write a grammar that generate the language:

$$L = \{w \in \{a,b,c\}^* \mid w \text{ does not contain } abc\}$$

$$
\begin{aligned}
\langle reset \rangle &\rightarrow b\langle reset \rangle \mid c\langle reset \rangle \\
&\quad \mid a\langle seenA \rangle \mid \lambda \\
\langle seenA \rangle &\rightarrow a\langle seenA \rangle \mid c\langle reset \rangle \\
&\quad \mid b\langle seenAB \rangle \mid \lambda \\
\langle seenAB \rangle &\rightarrow a\langle seenA \rangle \\
&\quad \mid b\langle reset \rangle \mid \lambda
\end{aligned}
$$

Context-free Grammars

Context-free Grammars

Definition

CFG = (V, T, P, S)

- V : Finite set of variables/non-terminals
- T : Alphabet/Finite set of terminals
- P : Finite set of rules/productions
- S : Start symbol

$$S \in V$$

$$V \cap T = \phi$$

$$Rule: \quad A \rightarrow \omega$$

$$A \in V \quad \omega \in (V \cup T)^*$$

Context-free Grammars

Definition

- *Context-freeness:* An *A*-rule can be applied whenever *A* occurs in a string, irrespective of the context (that is, non-terminals and terminals around *A*).

Context-free Grammars

Derivation

- One-step Derivation

$$uAv \underset{A \to \omega}{\Longrightarrow} u\omega v$$

- *w* is derivable from *v* in CFG, if there is a finite sequence of rule applications such that:

$$v \Rightarrow w_1 \Rightarrow \ldots \Rightarrow w_n = w$$

In this case we can write this derivation as *v* $\xrightarrow{*}$ *w*

Context-free Grammars

Derivation

The derivation as $v \Longrightarrow^{*} w$ is called:

Leftmost derivation: if in every step the leftmost variable is selected for reduction

Rightmost derivation: if in every step the rightmost variable is selected for reduction

Context-free Grammars

Example 1

Let $G = (\{S\}, \{a,b\}, S, P)$ with for P:

- $S \rightarrow aSa$, and $S \rightarrow bSb$, and $S \rightarrow \lambda$.
- Some *derivations* from this grammar:
- $S \Rightarrow aSa \Rightarrow aaSaa \Rightarrow aabSbaa \Rightarrow aabbaa$
- $S \Rightarrow bSb \Rightarrow baSab \Rightarrow baab$, and so on.

- In general $S \Rightarrow \ldots \Rightarrow ww^R$ for $w \in \{a,b\}^*$.

L(G)={$ww^R : w \in \{a,b\}^*$}

Context-free Grammars

Example 2

$$G = (\{S, A, B\}, \{a, b\},$$
$$\{S \to AB, A \to aA \mid \lambda,$$
$$B \to Bb \mid \lambda\},$$
$$S)$$

$$L(G) = L(a^* b^*)$$

Leftmost Derivation :

$$S \Rightarrow AB \Rightarrow aAB \Rightarrow aB \Rightarrow aBb \Rightarrow ab$$

Rightmost Derivation :

$$S \Rightarrow AB \Rightarrow ABb \Rightarrow Ab \Rightarrow aAb \Rightarrow ab$$

Context-free Grammars

Example 3

Take the CFG S → 0 | 1 | ¬(S) | (S)∨(S) | (S)∧(S), which generates all proper Boolean formulas that use "0", "1", "¬", "∨", "∧", "(" and ")".

Then "(0)∨((0)∧(1))" can be derived in the following ways…

[leftmost] S ⇒ (S)∨(S) ⇒ (0)∨(S) ⇒ (0)∨((S)∧(S))
⇒ (0)∨((0)∧(S)) ⇒ (0)∨((0)∧(1))

[rightmost] S ⇒ (S)∨(S) ⇒ (S)∨((S)∧(S)) ⇒ (S)∨((S)∧(1))
⇒ (S)∨((0)∧(1)) ⇒ (0)∨((0)∧(1))

[something else] S ⇒ (S)∨(S) ⇒ (0)∨(S) ⇒ (0)∨((S)∧(S))
⇒ (0)∨((S)∧(1)) ⇒ (0)∨((0)∧(0))

Context-free Grammars

Example 4

Consider the CFG:

$$G = \{\{S\},\{a,b\},\{S \rightarrow \lambda, S \rightarrow aSb\},S)$$

- Derivation of *aabb* is

$$S \Rightarrow aSb \Rightarrow aaSbb \Rightarrow aabb$$

Context-free Grammars

Example 5

Consider the CFG G:

$$S \rightarrow aSa \mid aBa$$

$$B \rightarrow bB \mid b$$

$$L(B) = \{b^m \mid m > 0\}$$

$$L(S) = \{a^n b^m a^n \mid n > 0 \wedge m > 0\}$$

L(G)= L(S)

Context-free Grammars

Example 6

Consider the CFG G_1:

$$S \to aSa \mid B$$
$$B \to bB \mid \lambda$$

The language generated by G_1 is:

$$L(G_1) = \{a^n b^m a^n \mid n \geq 0 \wedge m \geq 0\}$$

Consider the CFG G_2:

$$S \to abSc \mid \lambda$$

The language generated by G_2 is:

$$L(G_2) = \{(ab)^n c^n \mid n \geq 0\}$$

Context-free Grammars

Example 7

Consider the CFGs G_1 and G_2 :

G_1 :

$$S \rightarrow AB$$
$$A \rightarrow aA \mid a$$
$$B \rightarrow bB \mid \lambda$$

G_2 :

$$S \rightarrow aS \mid aB$$
$$B \rightarrow bB \mid \lambda$$

The language generated by G_1 and G_2 is:

$L(G_1) = L(G_2) = L(S)$

$$L(S) = \{a^n b^m \mid m \geq 0 \wedge n > 0\}$$
$$L(S) = L(a^+ b^*)$$

Context-free Grammars

Example 8

Write a CFG to generate the language: $a^*ba^*ba^*$

$$S \rightarrow AbAbA$$
$$A \rightarrow aA \mid \lambda$$

$$S \rightarrow aS \mid B$$
$$B \rightarrow bA$$
$$A \rightarrow aA \mid bC$$
$$C \rightarrow aC \mid \lambda$$

Left to right generation of string.

Context-free Grammars

Exercise 1

WRITE A CFG FOR THE EMPTY SET

$$G = \{ \{S\}, \Sigma, \emptyset, S \}$$

Context-free Grammars

Exercise 2

What is the CFG ({S},{(,)}, P, S) that produces
the language of correct parentheses like (), (()), or ()(())?

$S \rightarrow (S)|SS|\lambda$

Context-free Grammars

Example

Consider the CFG $G=(\{S,Z\},\{0,1\}, P, S)$ with

P: $S \rightarrow 0S1 \mid 0Z1$

$Z \rightarrow 0Z \mid \lambda$

What is the language generated by G?

Answer: $L(G) = \{0^i1^j \mid i \geq j \}$

Specifically, S yields the $0^{j+k}1^j$ according to:

$S \Rightarrow 0S1 \Rightarrow \ldots \Rightarrow 0^jS1^j \Rightarrow$

$0^jZ1^j \Rightarrow 0^j0Z1^j \Rightarrow \ldots \Rightarrow 0^{j+k}Z1^j \Rightarrow 0^{j+k}\varepsilon1^j = 0^{j+k}1^j$

Context-free Grammars

Exercise

- Can you make Context Free Grammars for the following?

a) $\{ 0^n1^n : n \geq 0\}$

b) $\{ 0^n1^m : n,m \geq 0\}$

c) Arithmetic a,b,c formulas like $a+b \times c+a$ (without ())

- Answers:

a) $S \rightarrow 0S1 \mid \lambda$

b) $S \rightarrow 0S \mid R$ and $R \rightarrow 1R \mid \lambda$

c) $S \rightarrow a \mid b \mid c \mid S+S \mid S \times S$

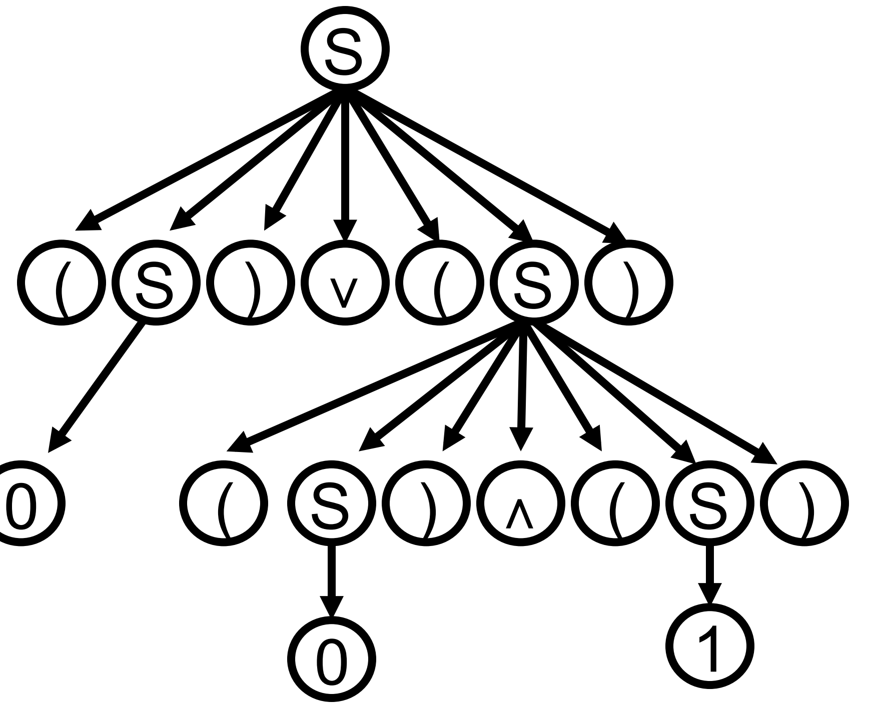
Context-free Grammars

Derivation Tree

For a CFG G=(V,T,S,P) a derivation tree has the following properties:

1) The root is labeled S
2) Each leaf is from $T \cup \{\lambda\}$
3) Each interior node is from V
4) If node has label $A \in V$ and its children $a_1 \ldots a_n$ (from L to R), then P must have the rule $A \rightarrow a_1 \ldots a_n$ (with $a_j \in V \cup T \cup \{\lambda\}$)
5) A leaf labeled λ is a single child (has no siblings).

Example



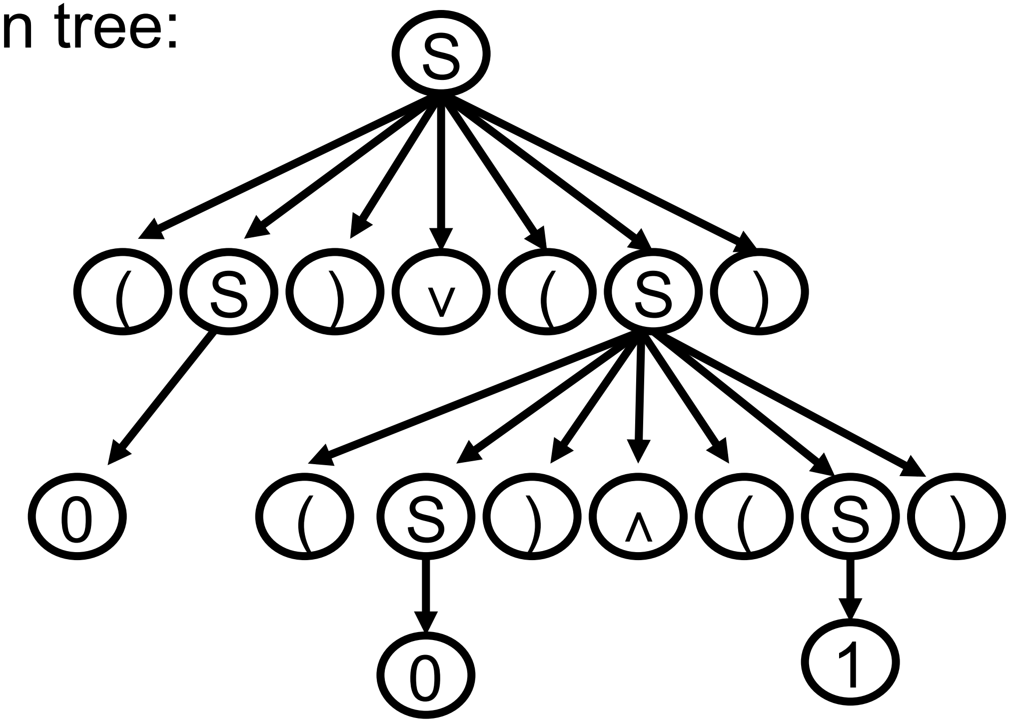
For **partial derivation trees** we have:
2a) Each leaf is from $V \cup T \cup \{\lambda\}$

Context-free Grammars

Derivation Tree: Example

Take the CFG $S \rightarrow 0 \mid 1 \mid \neg(S) \mid (S)\vee(S) \mid (S)\wedge(S)$, which generates all proper Boolean formulas that use "0", "1", "¬", "∨", "∧", "(" and ")".

The derivation $S \Rightarrow^* (0)\vee((0)\wedge(1))$ can be expressed by the following derivation tree:



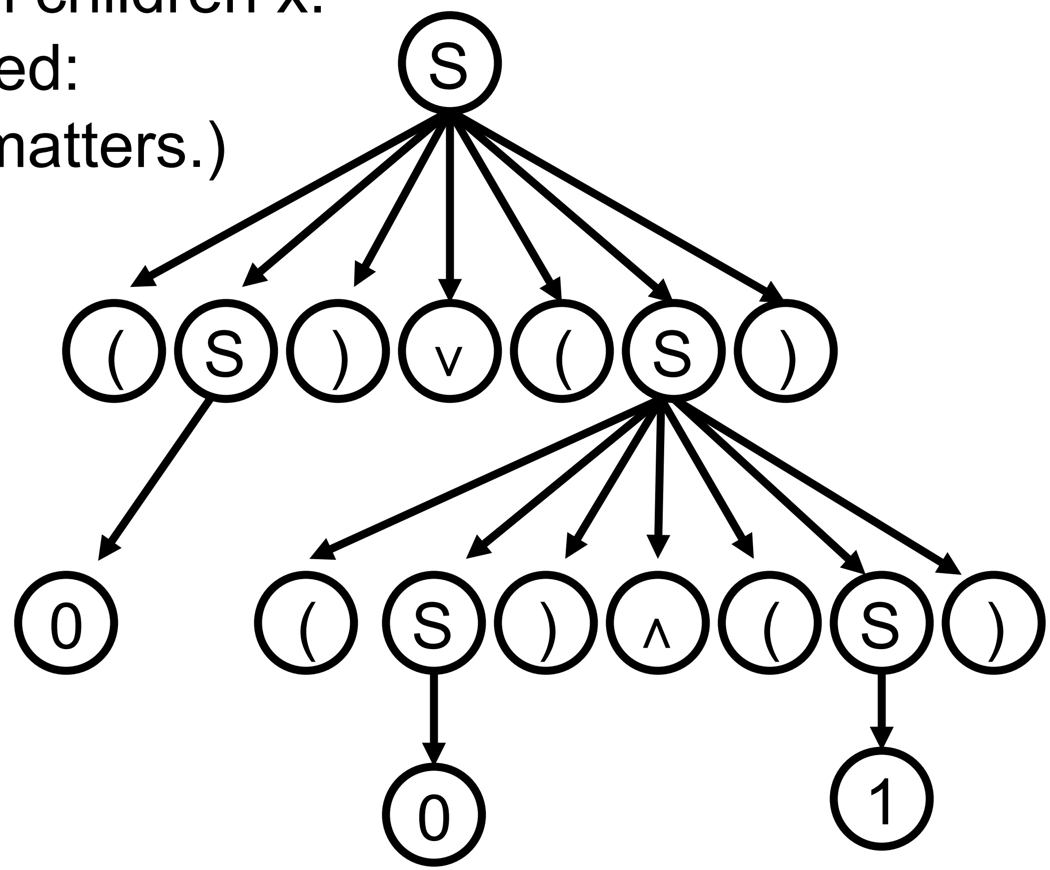
Context-free Grammars

Derivation Tree: Notes

Application of a production rule A → x is represented by node A with children x. (Note that the tree is ordered: the ordering of the nodes matters.)

The root has variable S.

The **yield** of S is expressed by the leaves of the tree.



Context-free Grammars

Derivation Tree: Notes

- Looking at a tree you see the derivation without the unnecessary information about its order.

- **Theorem**: Let G be a CFG. We have $w \in L(G)$ if and only if there exists a derivation tree of G with yield w.
- Also, y is a sentential form of G if and only if there exists a partial derivation tree for G.
- Remember: the root always has to be S.

Context-free Grammars

Example 1

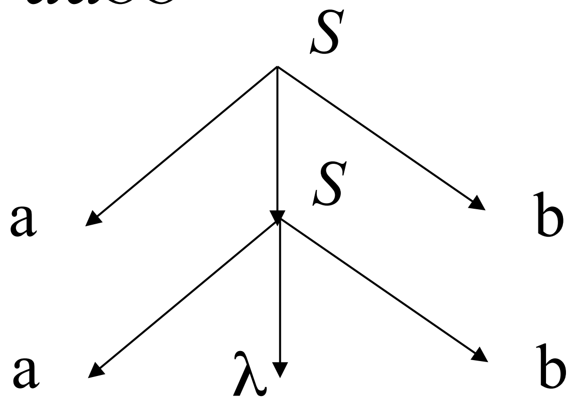
Consider the CFG G:

$$G = \{\{S\}, \{a, b\}, \{S \rightarrow \lambda, S \rightarrow aSb\}, S)$$

- The derivation of *aabb* is:

$$S \Rightarrow aSb \Rightarrow aaSbb \Rightarrow aabb$$

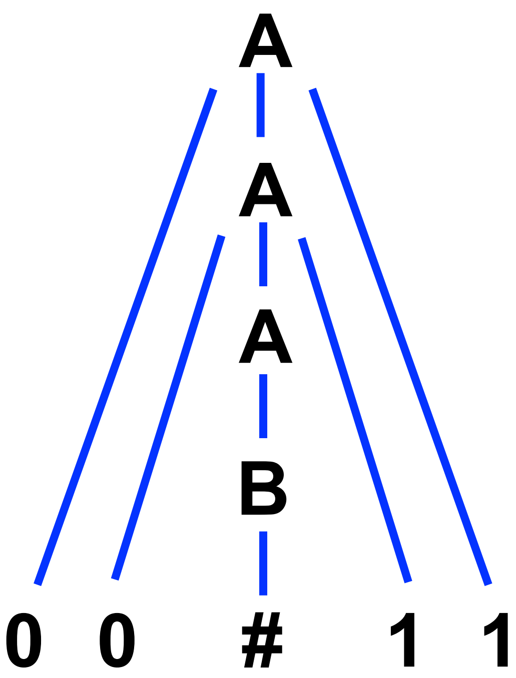
- Derivation tree is



Context-free Grammars

Example 2

$$A \Rightarrow 0A1 \Rightarrow 00A11 \Rightarrow 00B11 \Rightarrow 00\#11$$



Context-free Grammars

Example 3

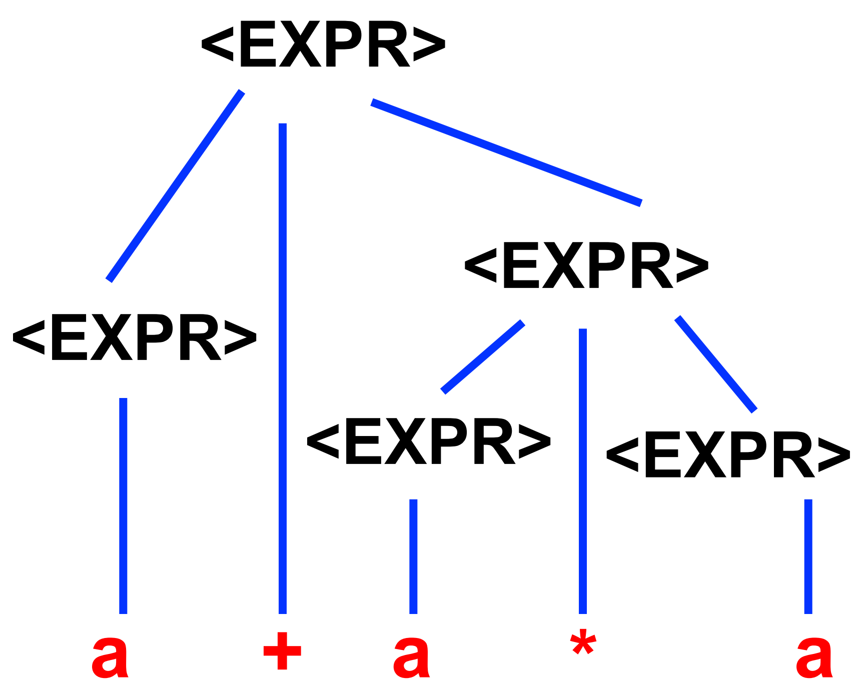
<EXPR> → <EXPR> + <EXPR>

<EXPR> → <EXPR> * <EXPR>

<EXPR> → (<EXPR>)

<EXPR> → a

Build a parse tree for a + a * a



Context-free Grammars

Exercise

WRITE A CFG FOR EVEN-LENGTH PALINDROMES over $\mathbf{\Sigma}$={a,b,c,d} ?

$S \rightarrow aSa$ for all $a \in \Sigma$

$S \rightarrow \lambda$