Automata and Languages

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Mathematical Background

- Sets
- Relations
- Functions
- Graphs
- Proof techniques

Functions

A function f is a binary relation (i.e. $f \subseteq A \times B$) written as $f : A \rightarrow B$ such that for $x \in A$ there exist at most one $y \in B$ for which $(x,y) \in f$, which we write as $f(x)=y$.

In other words: f is a function if

$f(x) = y$, $f(x)= z$ implies $y=z$

Example: $f: N \rightarrow N$ where $f(n)= n+1$ is a function on natural numbers.

Functions. Example

Example: Let $f : \mathbf{Z} \rightarrow \mathbf{R}$ be given by $f(x) = x^2$

Q1: What are the domain and co-domain? **(Z, R)**

Q2: What's the image of -3 ? **(9)**

Q4: What is the range $f(\mathbf{Z})$? **(set of perfect squares $f(\mathbf{Z}) = \{0,1,4,9,16,25,\ldots\}$)**

One-to-One, Onto, Bijection.

DEF: A function $f: A \rightarrow B$ is:

- ***one-to-one*** (or ***injective***) if different elements of *A* always result in different images in *B*.
  i.e. for all $a,b \in A$, $f(a)=f(b)$ implies $a=b$.
- ***onto*** (or ***surjective***) if the range of *f* is B ($f(A) = B$).
  i.e. for all $b \in B$, there exist $a \in A$ such that $f(a)=b$.
- a ***one-to-one correspondence*** (or a ***bijection***) if *f* is both one-to-one as well as onto.
  i.e. for all $b \in B$, there exist a unique $a \in A$ such that $f(a)=b$.

Quiz

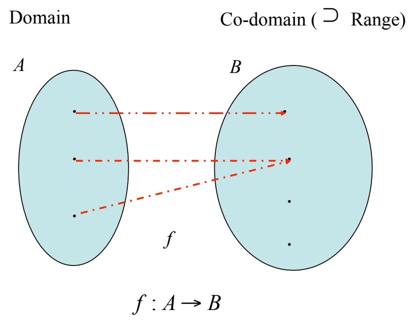
Q: Which of the following are 1-to-1, onto, a bijection?

1. $f: \mathbf{Z} \rightarrow \mathbf{R}$ is given by $f(x) = x^2$
2. $f: \mathbf{Z} \rightarrow \mathbf{R}$ is given by $f(x) = 2x$
3. $f: \mathbf{R} \rightarrow \mathbf{R}$ is given by $f(x) = x^3$
4. $f: \mathbf{Z} \rightarrow \mathbf{N}$ is given by $f(x) = |x|$
5. f : {people} → {people} is given by
   $f(x)$ = the father of x.

Answer

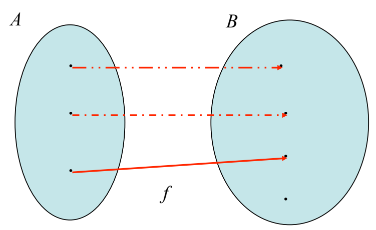
1. $f: \mathbf{Z} \rightarrow \mathbf{R}$, $f(x) = x^2$: none
2. $f: \mathbf{Z} \rightarrow \mathbf{R}$, $f(x) = 2x$: 1-1
3. $f: \mathbf{R} \rightarrow \mathbf{R}$, $f(x) = x^3$: 1-1, onto, bijection
4. $f: \mathbf{Z} \rightarrow \mathbf{N}$, $f(x) = |x|$: onto
5. $f(x)$ = the father of x : none

(Total) Function



One-One Function (injection)

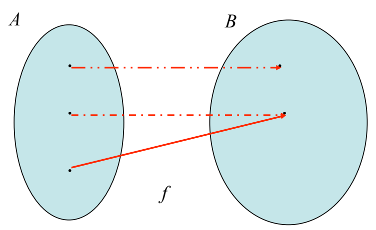
Domain Co-domain (⊃ Range)



$f : A \to B$

Onto Function (surjection)

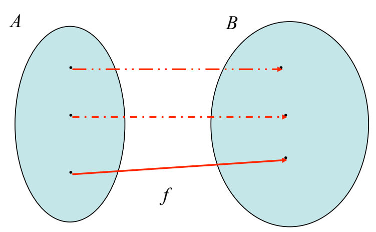
Domain Co-domain (= Range)



$f : A \to B$

One to one correspondence Function (bijection)

Domain Co-domain (= Range)



$f : A \to B$

Graphs

A ***graph*** *G* = (*V*,*E*) consists of a non-empty set *V* of ***vertices*** (or ***nodes***) and a set *E* (possibly empty) of ***edges*** where each edge is a subset of *V* with cardinality 2 (an unordered pair).

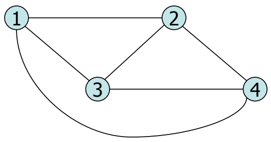
A **Path** is a sequence of vertices $v_1,\ldots,v_k$, $k \geq 1$, such that there exist an edge (v_i,v_{i+1}) for all $1 \leq i < k$.

Note that:

1. The length of such path is k-1
2. If $v_1=v_k$, the path is called a **cycle** path.

Graphs

Example



- (1)----(2)----(4) is a path
- (1)----(3)----(4)----(1) is a cycle

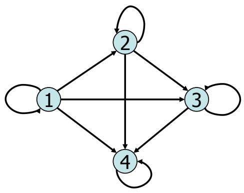
Digraphs

A ***directed graph*** (or ***digraph***) is a pair
$G = (V,E)$, where V is a non-empty set of
vertices (or ***nodes***) and E is a set of
arcs (ordered pairs of vertices) with $E \subseteq V \times V$.

- An arc (a,b) is denoted by $a \to b$
- A **Path** is a sequence of vertices $v_1,\dots,v_k, k \geq 1$, such that there exist an arc $v_i \to v_{i+1}$ for all $1 \leq i < k$.
- For an arc $v \to w$, v is called ***predecessor*** of w and w is called ***successor*** of v

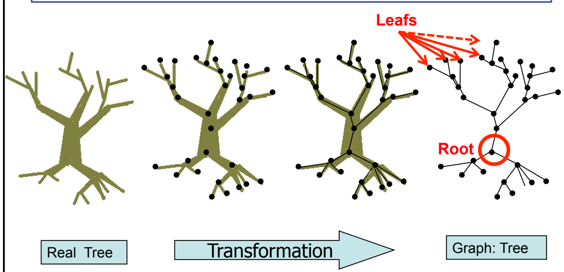
Digraphs

Example



Trees

A very important type of digraph in CS is called a *tree*:



Trees

Definition: A (ordered directed) tree is a digraph such that:

1. There exist one vertex called ***root*** with no predecessors and from which there exist a path to each other vertex

2. Each other vertex has exactly one predecessor

3. The successors of each vertex is ordered from left to right

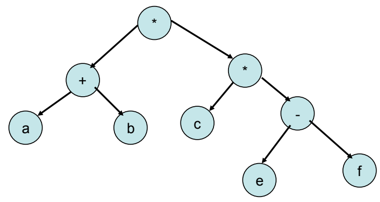
Conventions:
1. We draw trees with the root at the top and all arcs are pointing downward
2. The successors of each vertex will be drawn in left-to-right order

Terminology:
1. A successor of a vertex is called ***son***
2. The predecessor of a vertex is called a ***father***
3. For a path $v_1 \to \ldots \to v_n$, v_1 is called ancestor of v_n and v_n is called ***descendent*** of v_1
4. Any vertex is an ancestor and descendent of itself
5. A vertex with no sons is called ***leaf***, other vertices are called interior

Trees

Example

If *V={a,b,c,e,f,+,*,-)* and E is defined by the expression *(a+b)*(c*(e-f))*, then
We can draw the tree as follows:



Proof Techniques

•Proofs by Mathematical Induction

• Proofs by contradiction

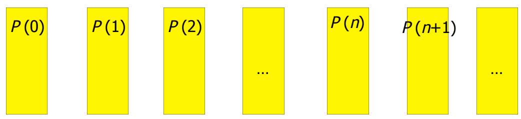
Mathematical Induction

Suppose we have a sequence of propositions which we would like to prove:
$P(0), P(1), P(2), P(3), P(4), \ldots P(n), \ldots$

For Example: $P(n)$: $\sum_{i=1}^{n}(2i-1) = n^2$

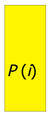
(The sum of the first n positive odd numbers is the n^{th} perfect square)

We can picture each proposition as a domino:



Mathematical Induction

When the domino falls (to right), the corresponding proposition P(i) is considered true:



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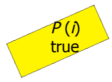
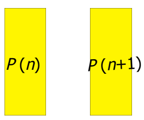


Mathematical Induction

Suppose that the dominos satisfy two conditions.

1) Well-positioned: If any domino falls (to right), next domino (to right) must fall also.



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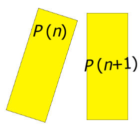

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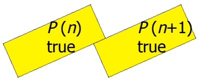
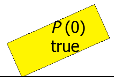
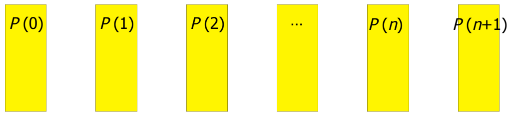


2) First domino has fallen to right



Mathematical Induction

Then can conclude that all the dominos fall!



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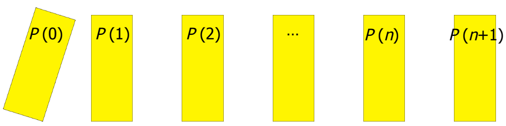

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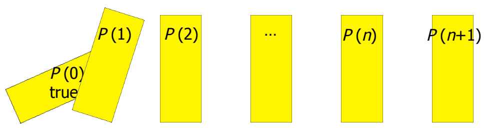

Mathematical Induction

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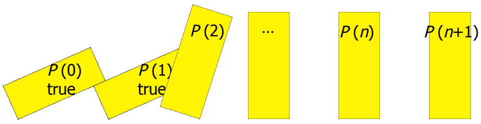

Mathematical Induction

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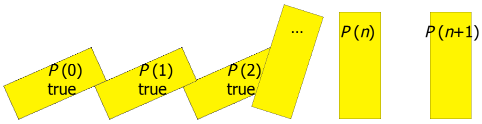

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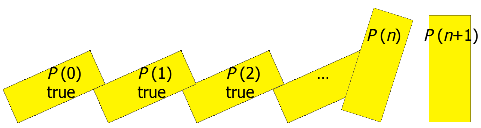

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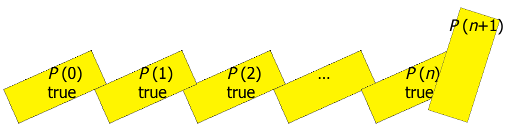

Mathematical Induction

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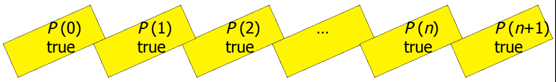
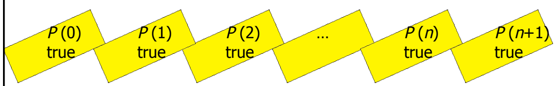


Mathematical Induction

Principle of Mathematical Induction:

If:	1. [**induction *basis***] *P* (0) is true 2. [***induction hypothesis & step***] ∀*n* *P*(*n*)→*P*(*n*+1) is true
Then:	∀*n* *P*(*n*) is true

This formalizes what occurred to dominos.



Mathematical Induction

In other words, what we need to do is:

1. **[induction *basis*]** Show that the statement P(n) holds for $n = 0$ (or whatever the smallest case is).
2. **[*induction hypothesis*]** Assume that P(n) is true
3. **[Induction *step*]** Show that *P*(*n*+1) is true

We then conclude that ∀*n* *P*(*n*) is true

Mathematical Induction: Example

Prove that $\forall n \geq 0$ $P(n)$ is true where

P (n): $\sum_{i=1}^{n} (2i-1) = n^2$

(The sum of the first *n* positive odd numbers is the *n*th perfect square)

We give two proofs:

1. Geometrical proof
2. Mathematical induction proof

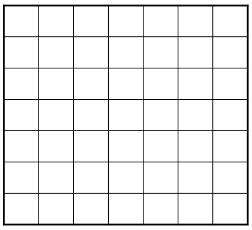
Mathematical Induction: Example

1. Geometrical proof

Geometric interpretation.

To get next square, need to add next odd number:

The sum of the first *n* positive odd numbers



the n^{th} perfect square

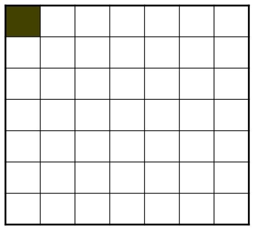
Mathematical Induction: Example

Geometric interpretation.

To get next square, need to add next odd number:

The sum of the first *n* positive odd numbers

1



$=1 = 1^2$

the n^{th} perfect square

Mathematical Induction: Example

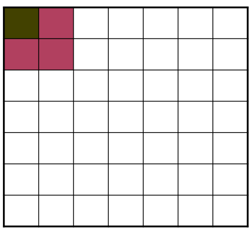
Geometric interpretation.

To get next square, need to add next odd number:

The sum of the first *n* positive odd numbers

1

+ 3



$=1 = 1^2$

$=4 = 2^2$

the n^{th} perfect square

Mathematical Induction: Example

Geometric interpretation.

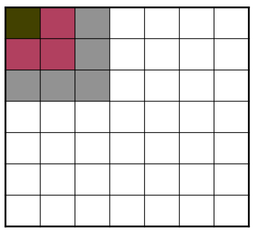
To get next square, need to add next odd number:

The sum of the first *n* positive odd numbers

1

+ 3

+ 5



$=1 = 1^2$

$=4 = 2^2$

$=9 = 3^2$

the n^{th} perfect square

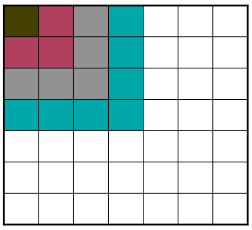
Mathematical Induction: Example

Geometric interpretation.

To get next square, need to add next odd number:

The sum of the first n positive odd numbers

1
+ 3
+ 5
+ 7



$=1 = 1^2$
$=4 = 2^2$
$=9 = 3^2$
$=16 = 4^2$

the n^{th} perfect square

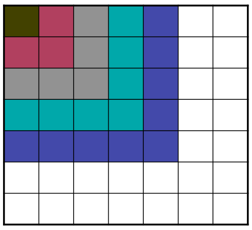
Mathematical Induction: Example

Geometric interpretation.

To get next square, need to add next odd number:

The sum of the first n positive odd numbers

1
+ 3
+ 5
+ 7
+ 9



$=1 = 1^2$
$=4 = 2^2$
$=9 = 3^2$
$=16 = 4^2$
$=25 = 5^2$

the n^{th} perfect square

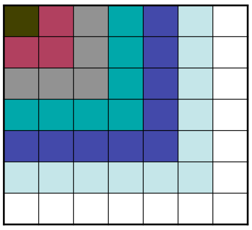
Mathematical Induction: Example

Geometric interpretation.

To get next square, need to add next odd number:

The sum of the first n positive odd numbers

1
+ 3
+ 5
+ 7
+ 9
+ 11



$=1 = 1^2$
$=4 = 2^2$
$=9 = 3^2$
$=16 = 4^2$
$=25 = 5^2$
$=36 = 6^2$

the n^{th} perfect square

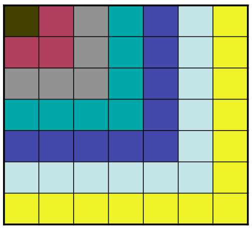
Mathematical Induction: Example

Geometric interpretation.

To get next square, need to add next odd number:

The sum of the first n positive odd numbers

1
+ 3
+ 5
+ 7
+ 9
+ 11
+ 13



$=1 = 1^2$
$=4 = 2^2$
$=9 = 3^2$
$=16 = 4^2$
$=25 = 5^2$
$=36 = 6^2$
$=49 = 7^2$

the n^{th} perfect square

Mathematical Induction: Example

2. Mathematical induction proof

1. Basis: we would like to show that: $\sum_{i=1}^{0}(2i-1) = 0^2$

This is obvious to see

2. Hypothesis: assume that P(n) is true, i.e. $\sum_{i=1}^{n}(2i-1) = n^2$

3. Step: show that P(n+1) is true

$$\sum_{i=1}^{n+1}(2i-1) = \sum_{i=1}^{n}(2i-1) + [2(n+1)-1]$$
$$= n^2 + [2n+1]$$
$$= (n+1)^2$$

Proof by Contradiction

To prove that some statement P is true we do:

1. Assume that P is false
2. Continue in the proof based on the nature of P
3. If we reach a wrong conclusion **(contradiction)** we conclude that P is true

Mathematical interpretation of Proof by Contradiction

For any k assume: $P(k) \wedge \neg Q(k)$

and derive: $\neg P(k) \vee Q(k)$

Uses the logical equivalence:

$P \rightarrow Q \Leftrightarrow \neg P \vee Q \Leftrightarrow \neg P \vee Q \vee \neg P \vee Q$

$\Leftrightarrow (\neg P \vee Q) \vee (\neg P \vee Q) \Leftrightarrow \neg(P \wedge \neg Q) \vee (\neg P \vee Q)$

$\Leftrightarrow (P \wedge \neg Q) \rightarrow (\neg P \vee Q)$

Intuitively: Assume claim is false (so *P* must be true and *Q* false). Show that assumption was absurd (so *P* false or *Q* true) so claim true!

Example

PROVE: The square of an even number k is even.

1. Assume that k^2 is not even.
2. So k^2 is odd.
3. $\exists n \; k^2 = 2n + 1$
4. $\exists n \; k^2 - 1 = 2n$
5. $\exists n \; (k-1)(k+1) = 2n$
6. Since $2n$ is even then: $(k-1)$ is even Or $(k+1)$ is even
7. $\exists a$ such that $k - 1 = 2a$ Or $\exists b$ such that $k+1 = 2b$
8. $\exists a$ such that $k = 2a + 1$ Or $\exists b$ such that $k = 2b - 1$
9. In both cases k is odd
10. Contradiction (with the fact that the given k is an even number)
11. Our assumption (k^2 is not even) is wrong
12. Hence k^2 is even

Exercise

The set of **rational** numbers

Q = { p/q | p,q are integers with no common factors and $q \neq 0$ }

Proof that the square root of 2 is NOT rational