Automata and Languages

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Mathematical Background

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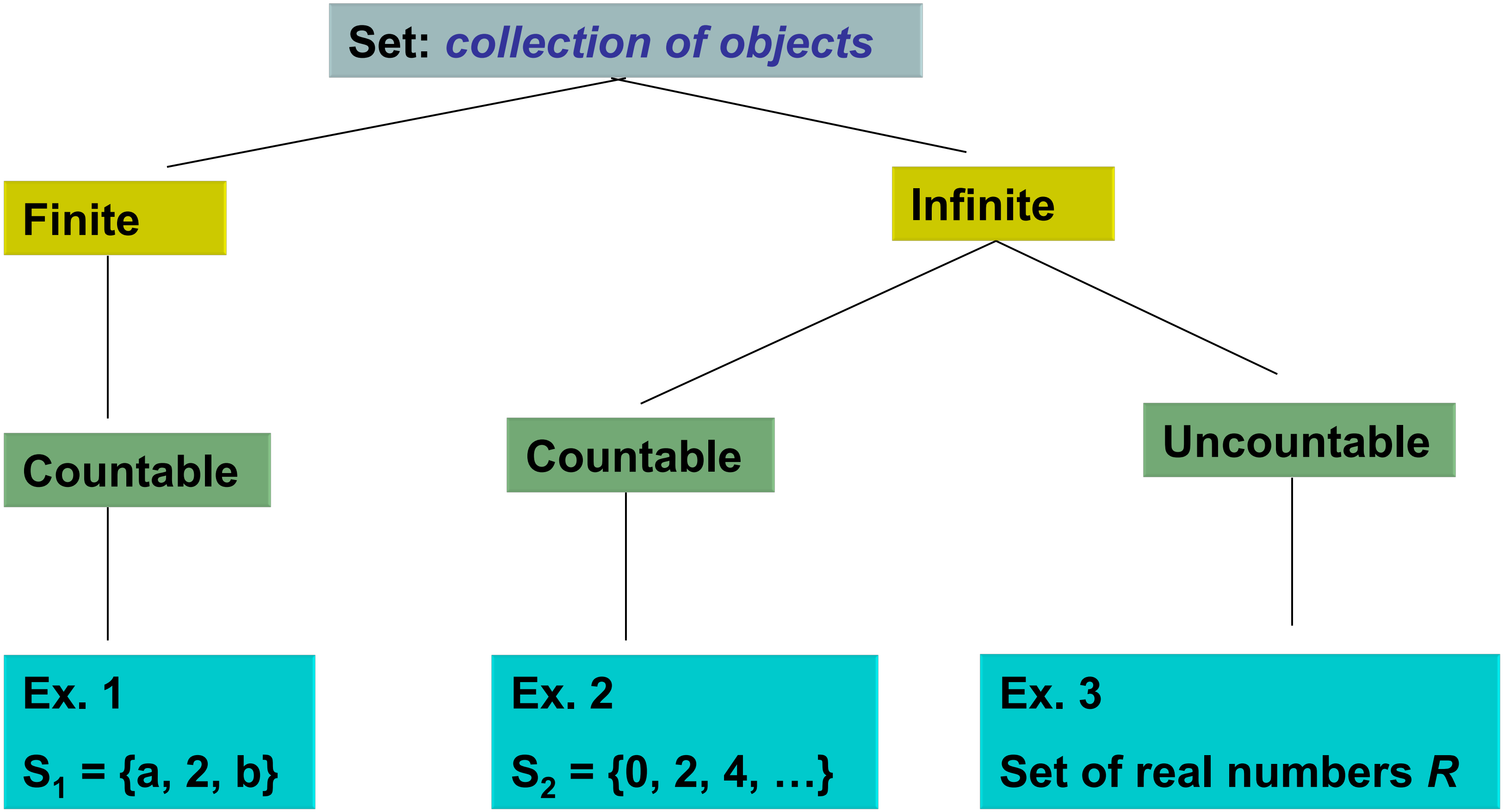
- Sets
- Relations
- Functions
- Graphs
- Proof techniques

Sets



Set: *collection of objects*
Finite
Infinite
Countable
Countable
Uncountable
Ex. 1
S_1 = {a, 2, b}
Ex. 2
S_2 = {0, 2, 4, …}
Ex. 3
Set of real numbers *R*

$\in$-Notation & $\subseteq$-Notation

The Greek letter **"$\in$"** (epsilon) is used to denote that an object is an *element* of a set. When crossed out **"$\notin$"** denotes that the object is *not an element*."

Ex.: $3 \in S$ reads:

"3 is an element of the set S ".

A set S is said to be a ***subset*** of the set T iff every element of S is also an element of T. This situation is denoted by

$$S \subseteq T$$

Specifying Sets

– By listing elements :

Ex.

- $\mathbf{S_1 = \{1, 2, 4, 5, 10, 20\}}$
- $\mathbf{S_2 = \{0, 2, 4, \ldots\}}$

– By defining property :

Ex.

- $\mathbf{S_1 = \{n : n} \in N$, n divides 20$\}$
- $\mathbf{S_2 = \{n : n} \in N$, n is even number$\}$

Examples

- { 11, 12, 13 }
- {  ,  ,  }
- {  ,  ,  , 11, Leo }
- { 11, 11, 11, 12, 13 } = { 11, 12, 13 } ?
- {  ,  ,  } = {  ,  ,  } ?

The Empty Set

The ***empty set*** is the set containing no elements. This set is also called the ***null set*** and is denoted by:

– { }

– $\emptyset$

Quiz

1. $\emptyset \subseteq \emptyset$? **(yes)**
2. $\emptyset \subset \emptyset$? **(No)**

Cardinality

The ***cardinality*** of a set is the number of distinct elements in the set. $|S|$ denotes the cardinality of S.

Q: Compute each cardinality.

1. $|\{1, -13, 4, -13, 1\}|$? **(3)**
2. $|\{3, \{1,2,3,4\}, \emptyset\}|$? **(3)**
3. $|\{\}|$? **(0)**
4. $|\{ \{\}, \{\{\}\}, \{\{\{\}\}\} \}|$? **(3)**

Set Theoretic Operations

Set theoretic operations allow us to build new sets out of old.

Given sets *A* and *B*, the set theoretic operators are:

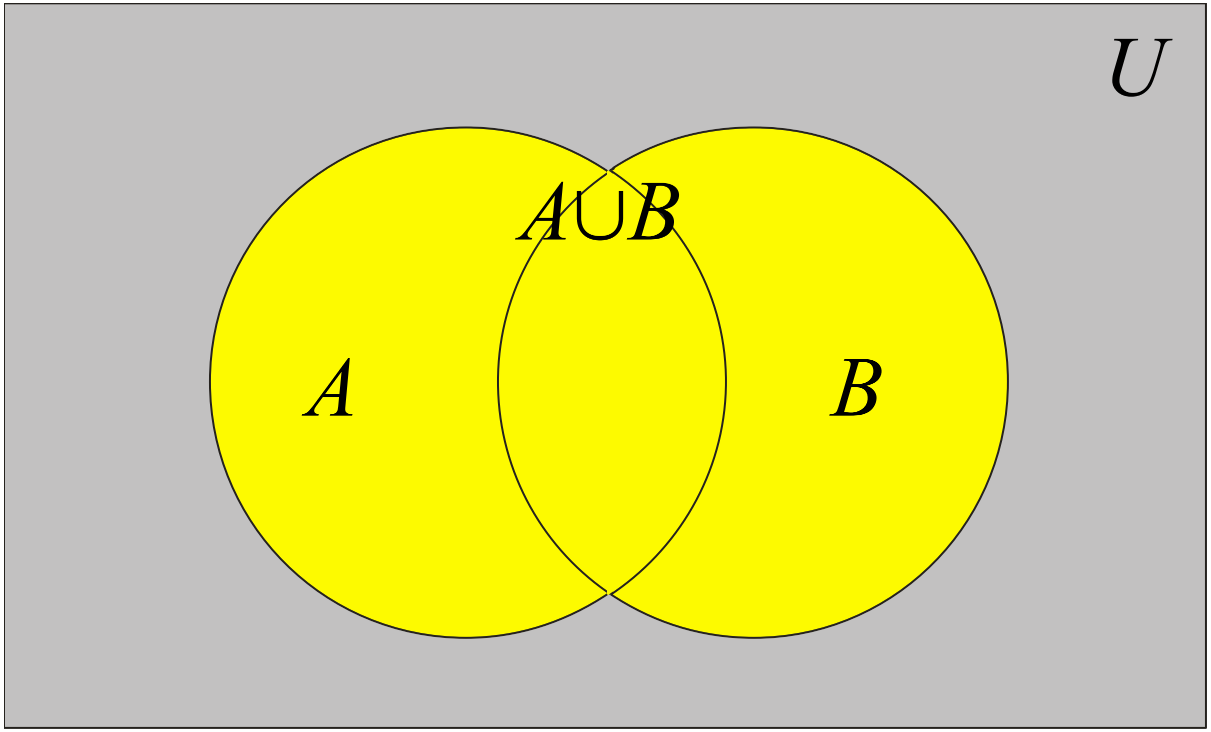
- Union ($\cup$)
- Intersection ($\cap$)
- Difference (-)
- Complement ("—")
- Cartesian Product: $A \times B$
- Power set: P(A)

give us new sets $A \cup B$, $A \cap B$, $A - B$, $\overline{A}$, AxB, and P(A).

Union

Elements in at least one of the two sets:

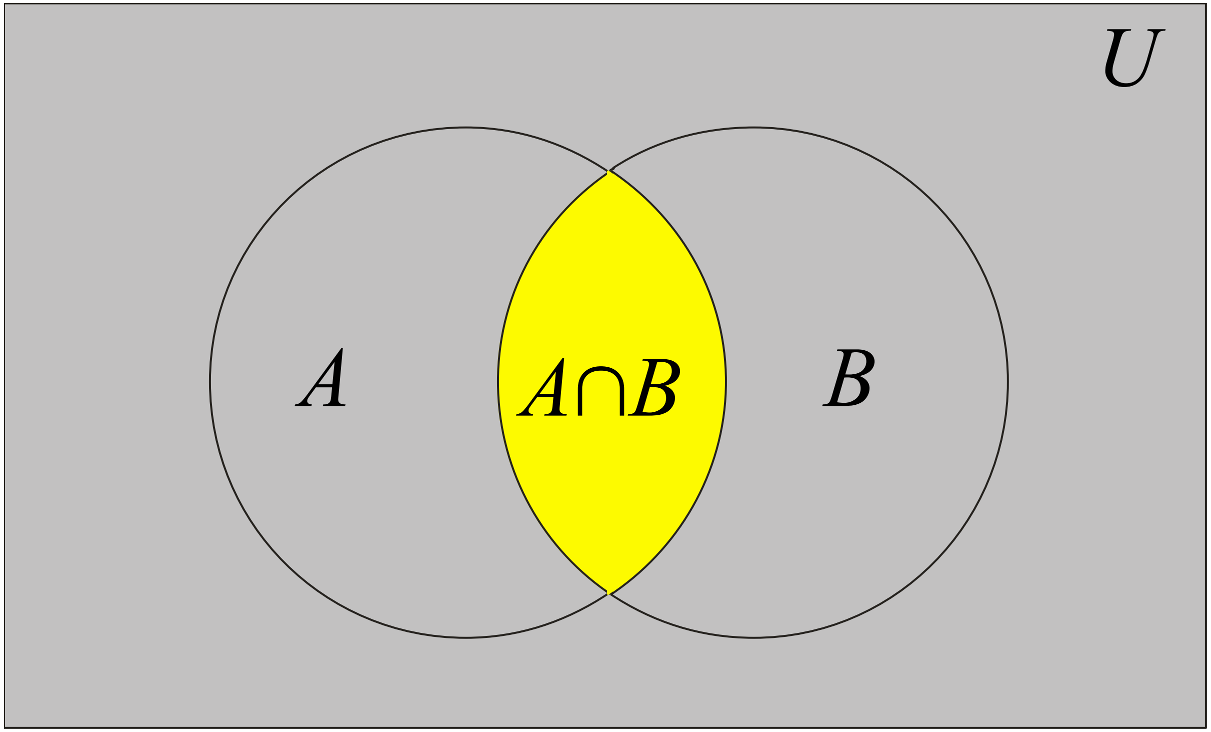
$$A \cup B = \{\, x \mid x \in A \ \vee \ x \in B \,\}$$



Intersection

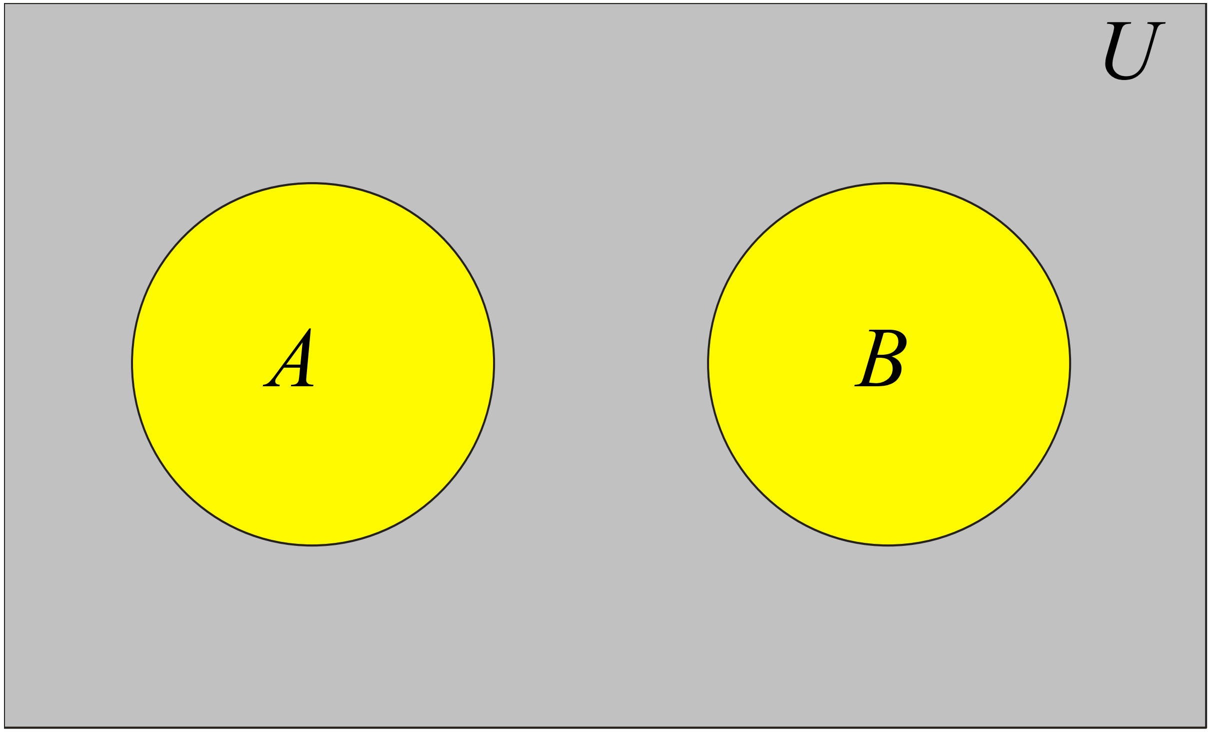
Elements in exactly one of the two sets:

$$A \cap B = \{\ x \mid x \in A\ \wedge\ x \in B\ \}$$



Disjoint Sets

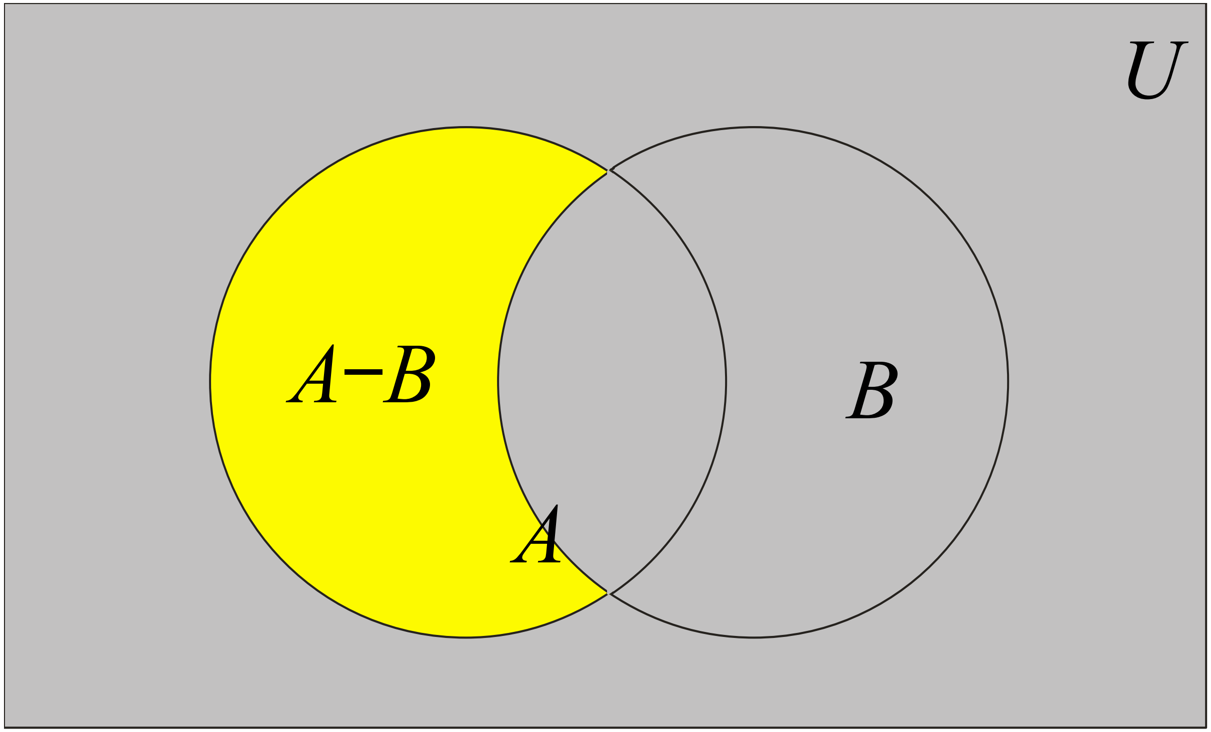
DEF: If A and B have no common elements, they are said to be ***disjoint***, i.e. $A \cap B = \varnothing$.



Set Difference

Elements in first set but not second:

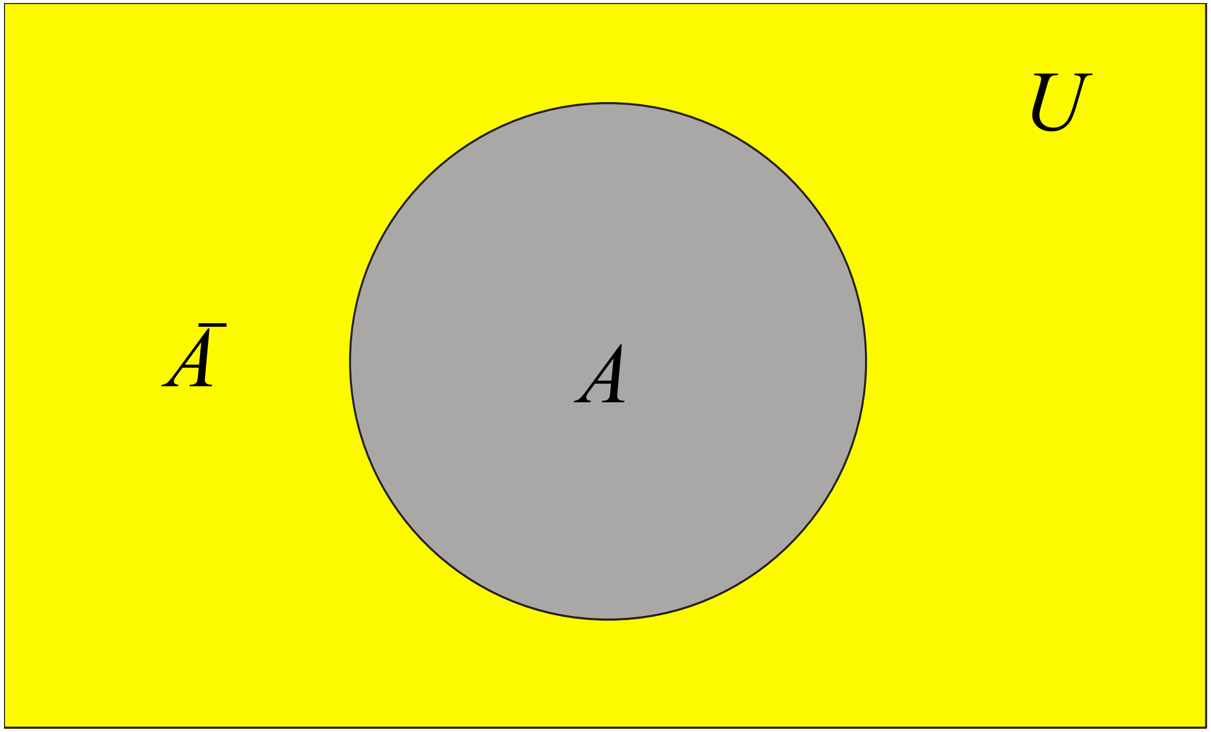
$$A - B = \{\, x \mid x \in A \;\wedge\; x \notin B \,\}$$



Complement

Elements not in the set (unary operator):

$$\bar{A} = \{\ x \mid x \notin A\ \}$$



Cartesian Product

The most famous example of 2-tuples are points in the Cartesian plane $\mathbf{R}^2$. Here ordered pairs (x,y) of elements of **R** describe the coordinates of each point. We can think of the first coordinate as the value on the x-axis and the second coordinate as the value on the y-axis.

The ***Cartesian product*** of two sets A and B (denoted by $A \times B$) is the set of all ordered pairs (a, b) where $a \in A$ and $b \in B$.

- $A \times B = \{ (x,y) \mid x \in A \text{ and } y \in B \}$

Q: What does $\varnothing \times S$ equal? (= $\varnothing$)

Some Examples

$L_{<6} = \{ x \mid x \in N, x<6 \}$

$L_{<6} \cap L_{prime} = \{2,3,5\}$

$\Sigma = \{0,1\}$

$\Sigma \times \Sigma = \{(0,0), (0,1), (1,0), (1,1)\}$

Power Sets

Formal: $P(A) = \{ S \mid S \subseteq A \}$

Example: $A = \{x,y\}$
$P(A) = \{ \{\} , \{x\} , \{y\} , \{x,y\} \}$

Note the different sizes:

$| P(A)| = 2^{|A|}$

$|A \times A| = |A|^2$

Power Sets

The ***power set*** of *S* is the set of all subsets of *S*.

Denote the power set by *P* (*S*) or by 2^S .

The latter weird notation comes from the following lemma.

Lemma: $| 2^S | = 2^{|S|}$

Power Sets: Example

To understand the previous fact consider

$$S = \{1,2,3\}$$

Enumerate all the subsets of S :

0-element sets: {} **1**

1-element sets: {1}, {2}, {3} **+ 3**

2-element sets: {1,2}, {1,3}, {2,3} **+ 3**

3-element sets: {1,2,3} **+ 1**

Therefore: $| 2^S | = 8 = 2^3 = 2^{|S|}$

Binary Relations

A binary relation R is a set of pairs of elements of sets A and B,

i.e. $R \subseteq A \times B$

- *A* is called the domain of R
- *B* is called the range (or codomain) of R
- If $A=B$ we say that R is a relation on *A*
- We may write *aRb* for $(a,b) \in R$

Properties of Binary Relations

Special properties for relation *on* a set A:

- ***reflexive*** : every element is self-related. i.e. aRa for all $a \in A$
- ***symmetric*** : order is irrelevant. i.e. for all $a,b \in A$ aRb iff bRa
- ***transitive*** : when a is related to b and b is related to c, it follows that a is related to c. i.e. for all $a,b,c \in A$ aRb and bRc implies aRc

Properties of Binary Relations

- ***asymmetric*** : also *not* equivalent to "not symmetric". Meaning: it's never the case that both *aRb* and *bRa* hold.

  i.e. $aRb \Rightarrow b \not R a$

- ***irreflexive*** : *not* equivalent to "not reflexive". Meaning: it's never the case that *aRa* holds.

  i.e. *for all a, aRa*

Properties of Binary Relations

An *equivalence relation* R is a relation on a set *A* which is reflexive, symmetric and transitive.

- **Generalizes the notion of "equals".**
- **R partitions A into disjoint nonempty equivalence classes, i.e., $A = A_1 \cup A_2 \cup \ldots$, such that**
  - **$A_i \cap A_j = \varnothing$, for all $i \neq j$**
  - **$a, b \in A_i \Rightarrow aRb$**
  - **$a \in A_i$, $b \in A_j$, $i \neq j \Rightarrow a \not R b$**
  - **A_i's are called equivalence classes and their number may be infinite**

Examples

- **Set of Natural numbers is partitioned by the relation**
  R={(i, j): i = j "mod 5"} into five "equivalence classes":
  { {0,5,10,…}, {1,6,11,…}, {2,7,12,…}, {3,8,13,…}, {4,9,14,…} }
- **"String length" can be used to partition the set of all bit strings.**
  { {},{0,1},{00,01,10,11},{000,…,111},… }

Closures of Relations

- If P is a set of properties of relations, the P-closure of a relation R is the smallest relation R' such that:
  1. $R \subseteq R'$
  2. $aR'b \rightarrow P((a, b))$ is true
  3. No more elements in R'

- Ex. The transitive-closure of a relation R, denoted by R^+ is defined by:
  1. $aRb \rightarrow (a, b) \in R^+$
  2. $(a, b) \in R^+, (b, c) \in R^+ \rightarrow (a, c) \in R^+$
  3. Only elements in (1) and (2) are in R^+

- Note that: the {reflexive, transitive}-closure of a relation R is denoted by R^*

Examples

- For the relation R = {(1,2), (2,2), (2,3)} on the set {1,2,3},
- R^+ and R^* are
- R^+ ={(1,2), (2,2), (2,3), (1, 3)} R + Transitive
- R^* ={(1,1), (1,2), (2,2), (2,3), (1, 3), (3,3)} R + Transitive + Reflexive