

Distance Function Approximation for Constructive Shape Modeling

Pierre-Alain FAYOLLE, Alexander PASKO, Benjamin SCHMITT

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Distributed Parallel Processing Laboratory
The University of Aizu
Tsuruga, Ikki-Machi, Aizu-Wakamatsu City
Fukushima, 965-8580 Japan

<p>Title:</p> <p>Distance Function Approximation for Constructive Shape Modeling</p>	
<p>Authors:</p> <p>Pierre-Alain FAYOLLE*, Alexander PASKO**, Benjamin SCHMITT***</p>	
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<p>Abstract:</p> <p>We introduce a smooth approximation of the min/max operations for maintaining an approximate signed distance function in constructive shape modeling. Min/max functions can provide exact distance function for the complex object if the initial primitives are defined by distance functions. However, min/max operations suffer from C^1 discontinuity, which may result in unexpected behavior when further operations are applied to the object. The smooth approximation proposed here provides guaranteed C^1 continuity of the resulting function. By introducing an additional bounding band, we also guarantee a fixed upper bound of the distance function error at any given point. As such approximations of min and max functions are acceptable for distance based modeling, we call the resulting defining functions of shapes by the term signed approximate real distance functions (SARDF), approximate min function can be called SARDF intersection, and approximate max function - SARDF union.</p>	
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Distributed Parallel Processing Laboratory
The University of Aizu
 Aizu-Wakamatsu
 Fukushima 965-8580
 Japan

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1 Introduction

The distance from a given point to a point set is defined as the minimum of distances between this point and any point of the set. Definitions of shape models by exact real distance functions are useful in different applications such as:

- constant-radius offsetting and blending operations [RR84];
- surface metamorphosis and smoothing [PT92];
- object reconstruction from a set of cross-sections [JC94];
- rendering using sphere tracing [Har96];
- generation of skeletal shape representation [ZKT98];
- heterogeneous object modeling [BST02].

The signed real distance function can define a closed surface as a zero value point set, and take positive values inside and negative values outside the surface. It is known that such a distance function can be C^1 discontinuous on some surfaces, curves, or points in 3D space. This discontinuity can cause unexpected results of further operations on the object such as blending, metamorphosis, and others. Deriving analytical expressions for a distance function for a given shape is a tedious work. Applying constructive modeling is one of the practical approaches to derive a distance function. Constructive modeling is based on successively applying set-theoretic and other operations to predefined shapes (primitives). If two point sets are defined as $f_1(X) \geq 0$ and $f_2(X) \geq 0$, where f_1 and f_2 are signed real distance functions, then the union of two objects can be defined as $f_3 = \max(f_1, f_2)$ and the intersection as $f_3 = \min(f_1, f_2)$ [Sab68], [Ric73]. In constructive modeling, *min/max* operations result in the exact distance function for the entire complex object, if the initial primitives are described by exact distance functions. However, *min/max* operations result in C^1 discontinuity by definition, as mentioned above.

There are several works on replacing *min/max* functions in constructive modeling by C^1 continuous exact or approximated descriptions of set-theoretic operations. R-functions [Rva63], [Rva74] provide distance properties of the defining function but not the exact distance function value. Moreover, exponential function value growth can be observed, for example, when applying R-functions to define union of a number of overlapping solids ("positive explosion" effect). A normalization procedure [Rva82], [BS01] can be applied to the resulting function to approximate the distance function. However, this recursive procedure is quite computationally expensive. Furthermore, the distance property of the resulting function is provided for points only close to the boundary. The superelliptic *min/max* approximation [Ric73] do not describe exact set-theoretic operations and suit for blending only. The elliptic *min/max* approximation [BDS⁺03], if applied to normalized primitives, can well approximate the distance near the boundary, but the error of the distance function grows infinitely far from the boundary.

We propose here to extend the approach of [BDS⁺03] for providing approximation of real distance functions by using a circular *min/max* approximation

and by introducing an additional bounding band to guarantee a fixed upper limit of the distance function error at any given point.

As such approximations of *min* and *max* functions are acceptable for distance-based modeling, we call the resulting defining functions of shapes by the term signed approximate real distance functions (SARDF), approximate *min* function can be called “SARDF intersection”, approximate *max* function - “SARDF union”.

2 Circular approximation of *min* and *max* functions

In this section, we introduce the circular approximation of the *min* and *max* functions for the set-theoretic operations to approximate signed real distance functions. Any contour line of the *min* and *max* functions has sharp corner, corresponding to the union of two vertical and horizontal rays, see for instance Fig. 1 showing the sharp corners appearing when drawing different contour lines for the *min* function. This feature of the contour lines reflects the C^1 discontinuity of the *min* and *max* functions that occurs at any point when two arguments are equal.

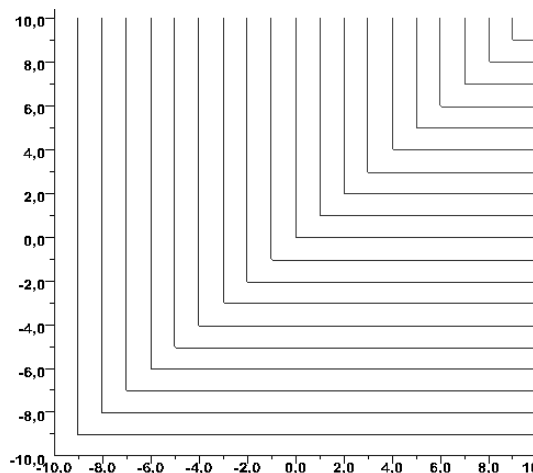


Figure 1: Each contour line of the *min* function contains a sharp corner.

Following the general approach of [BDS⁺03], we propose to replace this sharp corner in any contour line with a circular arc. All operations are discussed for two half spaces $f_1 = x$, $f_1 \geq 0$ and $f_2 = y$, $f_2 \geq 0$. We consider two straight lines, with the symmetry line defined by $y = x$ and with an angle θ between these lines, which act as a frontier for the circular arcs. Later we intend to apply these operations to arbitrary distance functions f_1 and f_2 .

The Euclidean plane is divided into four quadrants; the first quadrant corresponds to $x > 0$ and $y > 0$, the second quadrant to $x < 0$ and $y > 0$, the third quadrant to $x < 0$ and $y < 0$, and finally the fourth quadrant to $x > 0$ and $y < 0$. In the second and fourth quadrants, the approximate functions for *min* and *max* are equal exactly to *min* and *max*; thus we will restrict the discussion to the first and third quadrants, where the sharp corners need to be smoothed.

2.1 Circular *min* function approximation

2.1.1 Circular *min* approximation: quadrant I

We discuss here the circular approximation of the function $F(x, y) = \min(x, y)$ in the first quadrant, where $x > 0$ and $y > 0$. We want to replace any contour

lines $F = d$ with a circular arc and two rays tangentially attached to it as shown in Fig. 2. The angle θ made by two straight lines L_1 and L_2 is introduced as in Fig. 3.

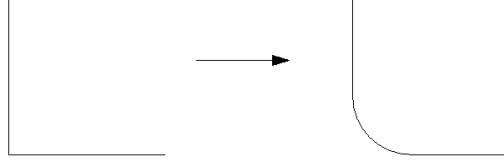


Figure 2: Contour line of the function \min has a sharp corner (left) to be replaced by a circular arc (right).

These two straight lines L_1 and L_2 break this first quadrant into three zones: A (below L_1), B (between L_1 and L_2) and C (above L_2), as shown in Fig. 3.

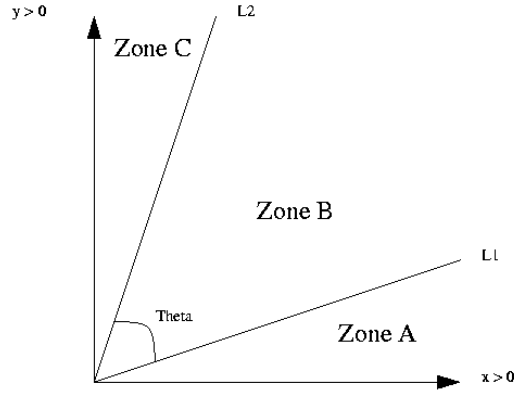


Figure 3: Two straight lines with the angle θ between them break the first quadrant into three zones.

The attachment points P_1 and P_2 of the arc and the rays are placed on the lines L_1 and L_2 correspondingly. Fig. 4 shows such a contour line configuration.

We are interested in the contour lines $\tilde{F} = d$ of the smooth approximation \tilde{F} of the \min function. Given an arbitrary point $P(x, y)$, we need to calculate a function value d for it.

In zone A, \tilde{F} is equal to $\min(x, y)$, therefore the contour is a horizontal line going through the point P and defined as $\tilde{F} = y$. In zone C, \tilde{F} is also equal to $\min(x, y)$, so the contour is a vertical line going through the point P and defined as $\tilde{F} = x$.

Finally, in zone B, we want to have a circular arc passing through the point $P(x, y)$. This arc should go through the point P and change into the horizontal ray in zone A and into the vertical ray in zone C. Both of these rays are at the distance d from the corresponding x and y axes. Such a distance is used for the definition of the value of the function. In order to calculate this distance d , we

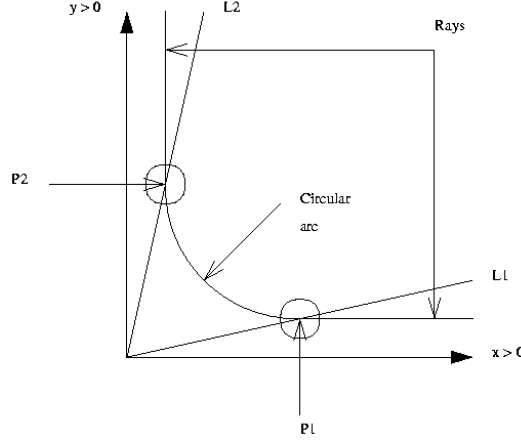


Figure 4: New contour line configuration: two frontier straight lines, two rays, arc, and two junction points between the rays and the arc located on the straight lines.

start from the equation of the circle passing through P :

$$(x - x_0)^2 + (y - y_0)^2 = R^2 \quad (1)$$

In this equation, x_0 , y_0 and R are unknown but can be expressed in terms of the value d being searched, and α , the angle between the straight lines and the axes. Fig. 5 shows the unknowns and their geometric relations.

First, α is expressed using θ (a parameter left to the user, expressing the angle between L_1 and L_2): $\alpha = \frac{(\frac{\pi}{2} - \theta)}{2}$. Then, from the lower triangle in zone A (Fig. 5), $x_0 = d \tan(\alpha)$. By analogy, from the upper triangle in zone C (Fig. 5), $y_0 = d \tan(\alpha)$, and $R = x_0 - d$. By replacing these variables in Eq. 1, we obtain the following quadratic equation for the variable d :

$$d^2 [\cotan^2(\alpha) + 2 \cotan(\alpha) - 1] - 2d(x + y) \cotan(\alpha) + x^2 + y^2 = 0 \quad (2)$$

The solution of the quadratic Eq. 2 for the unknown d is:

$$d = \begin{cases} \frac{-b \pm (b^2 - 4ac)^{0.5}}{2a} & \text{if } a \neq 0 \text{ and in zone B} \\ -\frac{c}{b} & \text{if } a = 0 \text{ and in zone B} \end{cases}$$

where $a = \cotan^2(\alpha) + 2 \cotan(\alpha) - 1$, $b = -2(x + y) \cotan(\alpha)$ and $c = x^2 + y^2$ are the coefficients of the quadratic Eq. 2.

The final expression for the value of \tilde{F} , at P in the quadrant I, is summarized below:

$$\tilde{F}(P) = d = \begin{cases} \frac{-b \pm (b^2 - 4ac)^{0.5}}{2a} & \text{if } a \neq 0 \text{ and } P \text{ in zone B} \\ -\frac{c}{b} & \text{if } a = 0 \text{ and } P \text{ in zone B} \\ y & \text{if } P \text{ in zone A} \\ x & \text{if } P \text{ in zone C} \end{cases}$$

where $a = \cotan^2(\alpha) + 2 \cotan(\alpha) - 1$, $b = -2(x + y) \cotan(\alpha)$ and $c = x^2 + y^2$, and α is an angle between L_1 and x -axis, and between L_2 and y -axis.

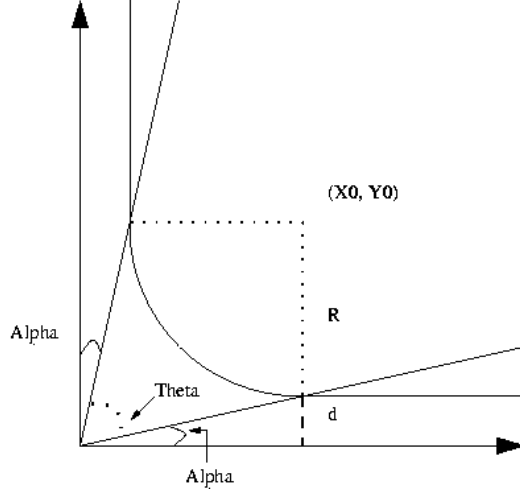


Figure 5: Unknowns of the Eq. 1 and their geometric relations.

2.1.2 Circular *min* approximation: quadrant III

We consider now the approximation \tilde{F} of the *min* function in the third quadrant, where $x < 0$ and $y < 0$. The method is the same as for the first quadrant. The continuation of the two straight lines L_1 and L_2 into the third quadrant, breaks it into three zones: D above L_1 , E between L_1 and L_2 , and F below L_2 as indicated in Fig. 6.

Given an arbitrary point $P(x, y)$ in the third quadrant, we want to evaluate \tilde{F} , the smooth approximation of the *min* function at this point.

In zone D, \tilde{F} is equal to $\min(x, y)$, therefore the contour is a vertical line going through the point P and defined as $\tilde{F} = x$. In zone F, \tilde{F} is also equal to $\min(x, y)$, so the contour is a horizontal line going through the point P and defined as $\tilde{F} = y$.

In zone E, we want to have a circular arc passing through the point $P(x, y)$. This arc should go through the point P and change into the horizontal ray in zone F and into the vertical ray in zone D. Both of these rays are at the distance d from the corresponding x and y axes. Such a distance is used for the definition of the value of the function.

Again, in order to calculate this distance d , we start from the equation of the circle passing through P :

$$(x - x_0)^2 + (y - y_0)^2 = R^2 \quad (3)$$

In this equation, x_0 , y_0 and R are unknown but can be expressed in terms of d and angle α . Fig. 7 shows the unknowns and their geometric relations.

First, α is expressed using θ (the angle between L_1 and L_2): $\alpha = \frac{(\frac{\pi}{2} - \theta)}{2}$. Then, from the triangle in zone D, $y_0 = -d \tan(\alpha)$. By analogy, from the triangle in zone F, $x_0 = -d \tan(\alpha)$, and $d = R + |x_0|$. By replacing these variables in Eq. 3, we obtain the quadratic equation for the variable d :

$$d^2 [\tan^2(\alpha) + 2 \tan(\alpha) - 1] + 2 d (x + y) \tan(\alpha) + x^2 + y^2 = 0 \quad (4)$$

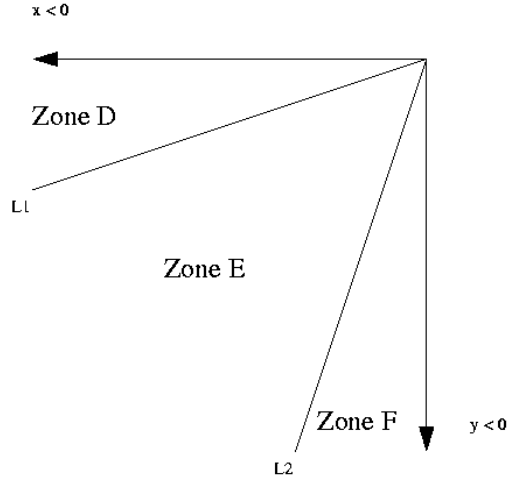


Figure 6: The continuation of L_1 and L_2 from the quadrant 1 ($x > 0$ and $y > 0$) breaks the quadrant 3 into three new zones D , E , and F .

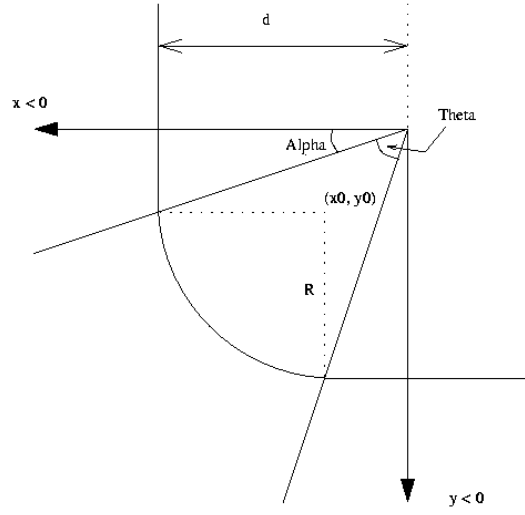


Figure 7: Unknowns of the Eq. 3 and their geometric relations.

The solution of the quadratic Eq. 4 for the unknown d is:

$$d = \begin{cases} -\frac{-b \pm (b^2 - 4ac)^{0.5}}{2a} & \text{if } a \neq 0 \text{ and } P \text{ in zone E} \\ \frac{c}{b} & \text{if } a = 0 \text{ and } P \text{ in zone E} \end{cases}$$

where $a = \tan^2(\alpha) + 2 \tan(\alpha) - 1$, $b = 2(x + y) \tan(\alpha)$ and $c = x^2 + y^2$ are the coefficients of the quadratic Eq. 4.

The final expression for the value of \tilde{F} , at P in the quadrant III, is summarized below:

$$\tilde{F}(P) = d = \begin{cases} -\frac{-b \pm (b^2 - 4ac)^{0.5}}{2a} & \text{if } a \neq 0 \text{ and } P \text{ in zone E} \\ \frac{c}{b} & \text{if } a = 0 \text{ and } P \text{ in zone E} \\ x & \text{if } P \text{ in zone D} \\ y & \text{if } P \text{ in zone F} \end{cases}$$

where $a = \tan^2(\alpha) + 2 \tan(\alpha) - 1$, $b = 2(x + y) \tan(\alpha)$ and $c = x^2 + y^2$.

2.2 Circular *max* function approximation

2.2.1 Circular *max* approximation: quadrant I

As for the *min* function, we want to replace the *max* function by a smooth approximation: the sharp corners of every contour line should be replaced by a circular arc as shown in Fig. 8.

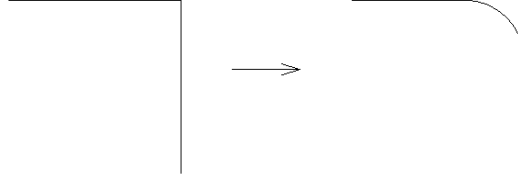


Figure 8: Sharp corner in the contour line of the *max* function shall be replaced by a circular arc.

Like in 2.1.1, two straight lines L_1 and L_2 are introduced; they break the first quadrant in three zones A , B and C , see Fig. 3. We are interested in the contour lines of the smooth approximation to the *max* function, \tilde{G} . Given an arbitrary point $P(x, y)$, we need to calculate a function value d for it.

In zone A, \tilde{G} is equal to $\max(x, y)$, therefore the contour is a vertical line going through the point P and defined as $\tilde{G} = x$. In zone C, \tilde{G} is also equal to $\max(x, y)$, so the contour is a horizontal line going through the point P and defined as $\tilde{G} = y$.

Finally, in zone B, we want to have a circular arc passing through the point $P(x, y)$. This arc should go through the point P and change into the vertical ray in zone A and into the horizontal ray in zone C. Both of these rays are at the distance d from the corresponding x and y axes. Such a distance is used for the definition of the value of the function.

In order to calculate this distance d , we start from the equation of the circle passing through P :

$$(x - x_0)^2 + (y - y_0)^2 = R^2 \quad (5)$$

In this equation, x_0 , y_0 and R are unknown but can be expressed in terms of d and the angle between the straight lines and the axes. Fig. 9 shows the unknowns and their geometric relations.

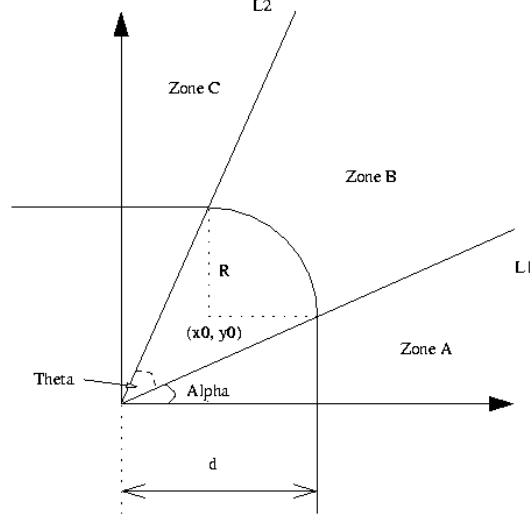


Figure 9: Unknowns of the Eq. 5 and their geometric relations.

First, α is expressed using θ (the angle between L_1 and L_2): $\alpha = \frac{(\frac{\pi}{2} - \theta)}{2}$. Then, from the lower triangle in zone A (Fig. 9), $y_0 = d \tan(\alpha)$. By analogy, from the upper triangle in zone C (Fig. 9), $x_0 = d \tan(\alpha)$. And, $R = d - x_0$. By replacing these variables in Eq. 5, we obtain a quadratic equation of the variable d :

$$d^2 [\tan^2(\alpha) + 2 \tan(\alpha) - 1] - 2 d (x + y) \tan(\alpha) + x^2 + y^2 = 0 \quad (6)$$

The solution of the quadratic Eq. 6 for the unknown d is:

$$d = \begin{cases} \frac{-b \pm (b^2 - 4ac)^{0.5}}{2a} & \text{if } a \neq 0 \text{ and in zone B} \\ -\frac{c}{b} & \text{if } a = 0 \text{ and in zone B} \end{cases}$$

where $a = \tan^2(\alpha) + 2 \tan(\alpha) - 1$, $b = -2 (x + y) \tan(\alpha)$ and $c = x^2 + y^2$ are the coefficients of the quadratic Eq. 6.

The final expression for the value of \tilde{G} , at P in the quadrant I, is summarized below:

$$\tilde{G}(P) = d = \begin{cases} \frac{-b \pm (b^2 - 4ac)^{0.5}}{2a} & \text{if } a \neq 0 \text{ and P in zone B} \\ -\frac{c}{b} & \text{if } a = 0 \text{ and P in zone B} \\ x & \text{if P in zone A} \\ y & \text{if P in zone C} \end{cases}$$

where $a = \tan^2(\alpha) + 2 \tan(\alpha) - 1$, $b = -2 (x + y) \tan(\alpha)$ and $c = x^2 + y^2$.

2.2.2 Circular \max approximation: quadrant III

Following the same strategy, we consider now the approximation \tilde{G} of the \max function in the third quadrant, where $x < 0$ and $y < 0$. The continuation of the two straight lines L_1 and L_2 into the third quadrant, breaks it into three zones: D above L_1 , E between L_1 and L_2 , and F below L_2 as indicated in Fig. 6.

Given an arbitrary point $P(x, y)$ in the third quadrant, we want to evaluate \tilde{G} , the smooth approximation of \max at that point.

In zone D, \tilde{G} has to repeat $\max(x, y)$, therefore we draw an horizontal line through the point P and take $\tilde{G} = y$. In zone F, \tilde{G} has to repeat $\max(x, y)$, so we draw a vertical line through the point P and take $\tilde{G} = x$.

In zone E finally, we want to have an arc of a circle passing through the point $P(x, y)$. This arc of circle should go through the point P and change into the vertical ray in zone F and into the horizontal ray in zone D. Both of these rays are at the distance d from the corresponding x and y axis. Such a distance is used for the definition of the value of the function.

Again, in order to calculate this distance d , we start from the equation of the circle passing through P :

$$(x - x_0)^2 + (y - y_0)^2 = R^2 \quad (7)$$

In this equation, x_0 , y_0 and R are unknown but can be expressed in terms of d , the searched value and α , the angle between the straight lines and the axes. Fig. 10 shows the unknowns and their geometric relations.

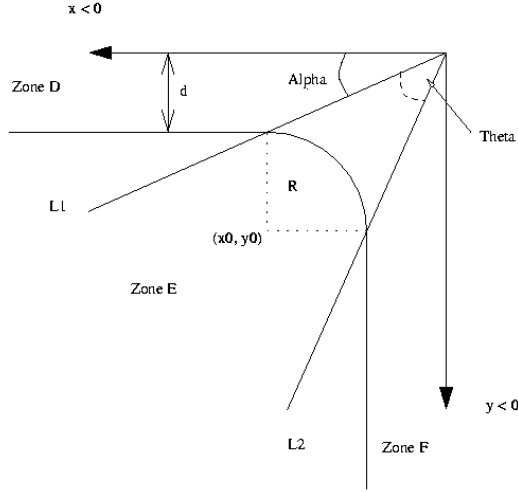


Figure 10: Unknowns of Eq. 7 and their geometric relations.

First, α is expressed using θ (the angle between L_1 and L_2): $\alpha = \frac{(\frac{\pi}{2} - \theta)}{2}$. Then, from the triangle in zone D, $y_0 = -d \tan(\alpha)$. By analogy, with the triangle, in zone F, $x_0 = -d \tan(\alpha)$. And finally, $R = |x_0| - d$. By replacing these variables in Eq. 7, we obtain a quadratic equation of the variable d :

$$d^2 [\cotan^2(\alpha) + 2 \cotan(\alpha) - 1] + 2 d (x + y) \cotan(\alpha) + x^2 + y^2 = 0 \quad (8)$$

The solution of the quadratic Eq. 8 for the unknown d is:

$$d = \begin{cases} -\frac{-b \pm (b^2 - 4ac)^{0.5}}{2a} & \text{if } a \neq 0 \text{ and } P \text{ in zone E} \\ \frac{c}{b} & \text{if } a = 0 \text{ and } P \text{ in zone E} \end{cases}$$

where $a = \cotan^2(\alpha) + 2 \cotan(\alpha) - 1$, $b = 2(x + y) \cotan(\alpha)$ and $c = x^2 + y^2$ are the coefficient of the quadratic Eq. 8.

The final expression for the value of \tilde{G} , at P in the quadrant III, is summarized below:

$$\tilde{G}(P) = d = \begin{cases} -\frac{-b \pm (b^2 - 4ac)^{0.5}}{2a} & \text{if } a \neq 0 \text{ and } P \text{ in zone E} \\ \frac{c}{b} & \text{if } a = 0 \text{ and } P \text{ in zone E} \\ y & \text{if } P \text{ in zone D} \\ x & \text{if } P \text{ in zone F} \end{cases}$$

where $a = \cotan^2(\alpha) + 2 \cotan(\alpha) - 1$, $b = 2(x + y) \cotan(\alpha)$ and $c = x^2 + y^2$.

2.3 Problems of the circular approximation

The use of the described above circular approximations for the *min* and *max* functions can provide the C^1 approximation of the resulting distance function for constructive shapes built using normalized primitives (defined by distance functions). Unfortunately, this approach has the following problem: the radius of the circular arc used to replace the sharp corners in the contour lines keeps growing with the distance from the initial surfaces. Fig. 11 illustrates this problem for the case of the *min* function approximation. Because of this behavior of the arc radius, the error of the distance function approximation grows infinitely with the distance, which is unacceptable for distance-based modeling and application algorithms.

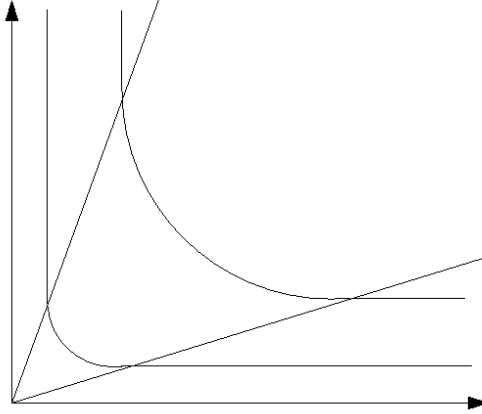


Figure 11: The radius of the circular arc is growing with the distance from the origin, thus increasing the error in the distance function made by the approximation.

We propose to prevent the radius from growing infinitely by introducing a fixed radius circular arc, and by switching to it, when some threshold for the radius is reached.

3 Approximation with additional bounding band

In the previous approach, the smoothing area grows infinitely in size inside the given angle. In this area in the case of *min* function, the exact distance is replaced by smaller approximate values with the growing error. It is better to introduce a fixed threshold for the radius R of the circular arc. A new bounding band can be introduced by two parallel straight lines that enclose the arcs with the fixed radius. These band lines are defined by a shift of the line $y = x$ at R distance in positive and negative x directions: $y = x - R$ and $y = x + R$. This measure will guarantee a known upper limit at any point in space for the distance error. As such approximations of *min* and *max* functions are acceptable for distance-based modeling, we call the resulting defining functions of shapes by the term signed approximate real distance functions (SARDF), approximate *min* function can be called "SARDF intersection", approximate *max* function - "SARDF union".

3.1 SARDF intersection

3.1.1 Approximate *min*: quadrant I

The intersections of the two parallel bounding band lines, $y = x + R$ and $y = x - R$, with the lines parallel to the axes, $x = R$ and $y = R$, result in two points: $A_1(2R, R)$ and $A_2(R, 2R)$, as shown in Fig. 12. These points are connected by the circular arc $(x - 2R)^2 + (y - 2R)^2 = R^2$. This makes a natural boundary, that splits the first quarter into two zones I and II, for applying two approaches to approximation (see Fig. 12).

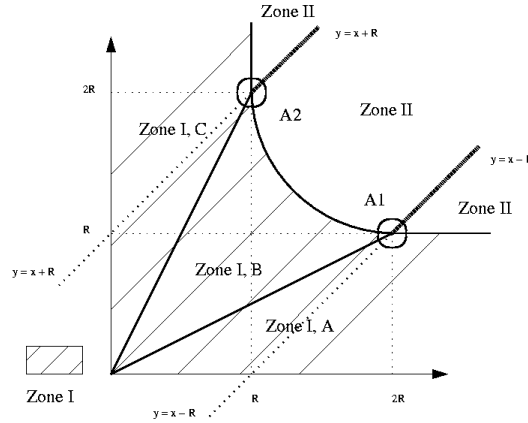


Figure 12: The first quadrant is divided into two zones. The circular approximation is applied in zone I, whereas we introduce a fixed radius approximation with the bounding band in zone II.

Zone I Zone I corresponds to the set of points $P(x, y)$ for which $x < R$ or $y < R$ or $(x < 2R$ and $y < 2R$ and $(x - 2R)^2 + (y - 2R)^2 > R^2$), see Fig. 12. In this zone, the circular approximation described in 2.1.1 is used. In this case, the bounding lines for the angle θ are:

- L_1 : $O - A_1$, where O is the origin $(0,0)$,
- L_2 : $O - A_2$.

and the angle α is defined by $\cotan(\alpha) = 2$. We remind the expression of the circular approximation $\tilde{F} = d$ for the *min* function, in the first quadrant:

$$\tilde{F}(P) = d = \begin{cases} \frac{-b \pm (b^2 - 4ac)^{0.5}}{2a} & \text{if } a \neq 0 \text{ and P in zone B} \\ -\frac{c}{b} & \text{if } a = 0 \text{ and P in zone B} \\ y & \text{if P in zone A} \\ x & \text{if P in zone C} \end{cases}$$

where $a = \cotan^2(\alpha) + 2 \cotan(\alpha) - 1 = 7$, $b = -2(x + y) \cotan(\alpha) = -4(x + y)$ and $c = x^2 + y^2$.

Zone II In the zone II, the fixed radius with the bounding band is applied to get a smooth approximation \tilde{F} of the *min* function. Outside the bounding band lines $y = x + R$ and $y = x - R$, the approximation of *min* is *min* itself, therefore we take $\tilde{F} = \min(x, y)$. Between the lines, we start from the equation of the circle: $(x - x_0)^2 + (y - y_0)^2 = R^2$, where x_0, y_0 are parameters that can be expressed using the radius R and the distance d , which is the searched value of \tilde{F} .

Fig. 13 displays these different parameters and their geometric relations. It is obvious that $x_0 = y_0 = d + R$. Replacing x_0 and y_0 in the equation of the circle, the following quadratic expression of the unknown d is obtained: $2d^2 + d(-2x - 2y + 4R) + (x^2 + y^2 - 2xR - 2yR + R^2) = 0$. The solution for the unknown d , gives the searched value for the smooth approximation of the *min* function: $d = \frac{-b \pm (b^2 - 4ac)^{0.5}}{2a}$, where $a = 2$, $b = -2x - 2y + 4R$ and $c = x^2 + y^2 - 2xR - 2yR + R^2$ are the coefficients of the quadratic equation.

The expression for \tilde{F} in the zone II becomes:

$$\tilde{F}(P) = d = \begin{cases} \frac{-b \pm (b^2 - 4ac)^{0.5}}{2a} & \text{inside the bounding band} \\ y & \text{below } y = x - R \\ x & \text{above } y = x + R \end{cases}$$

where $a = 2$, $b = -2x - 2y + 4R$ and $c = x^2 + y^2 - 2xR - 2yR + R^2$.

Final expression for the SARDF intersection in the quadrant I We give the final expression for the SARDF intersection \tilde{F} approximating the *min* function in the first quadrant. Given a point $P(x, y)$ in the first quadrant, the

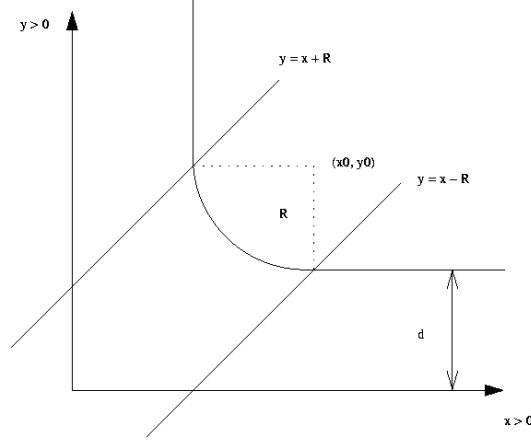


Figure 13: Geometric relations between the parameters of the circular approximation with the bounding band in zone II.

value d of \tilde{F} at P is given by:

$$\tilde{F}(P) = d = \begin{cases} \frac{-b_1 \pm (b_1^2 - 4a_1c_1)^{0.5}}{2a_1} & \text{if } a_1 \neq 0 \text{ and } P \text{ in zone I, B} \\ -\frac{c_1}{b_1} & \text{if } a_1 = 0 \text{ and } P \text{ in zone I, B} \\ y & \text{if } P \text{ in zone I, A} \\ x & \text{if } P \text{ in zone I, C} \\ \frac{-b_2 \pm (b_2^2 - 4a_2c_2)^{0.5}}{2a_2} & \text{if } P \text{ in zone II, and inside the bounding band} \\ y & \text{if } P \text{ in zone II and below } y = x - R \\ x & \text{if } P \text{ in zone II and above } y = x + R \end{cases}$$

where $a_1 = \cotan^2(\alpha) + 2 \cotan(\alpha) - 1 = 7$, $b_1 = -2(x + y) \cotan(\alpha) = -4(x + y)$, $c_1 = x^2 + y^2$, $a_2 = 2$, $b_2 = -2x - 2y + 4R$ and $c_2 = x^2 + y^2 - 2xR - 2yR + R^2$.

3.1.2 Approximate *min*: quadrant III

We consider now the same approach to approximation of the *min* function with the bounding band in the third quadrant. The intersections of the two parallel bounding band lines, $y = x + R$ and $y = x - R$, with the lines parallel to the axes $x = -R$ and $y = -R$, result in two points: $A_3(-2R, -R)$ and $A_4(-R, -2R)$, as shown in Fig. 14. These points are connected by the circular arc $(x + R)^2 + (y + R)^2 = R^2$. This makes a natural boundary, that splits the third quarter into two zones III and IV, for applying the two different approaches to approximation (see Fig. 14).

Zone III Zone III corresponds to the set of points $P(x, y)$ for which $x > -R$ or $y > -R$ or $(x > -2R$ and $y > -2R$ and $(x + R)^2 + (y + R)^2 < R^2$), see Fig. 14. In this zone, the circular approximation described above is applied. In this case, the bounding lines for the angle θ are:

- L_1 : $O - A_4$, where O is the origin $(0,0)$,

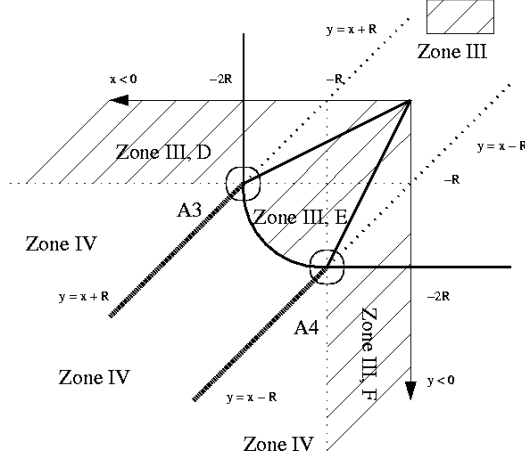


Figure 14: The third quadrant is divided into two zones. The circular approximation is applied in zone III, whereas we introduce a fixed radius approximation with the bounding band in zone IV.

- L_2 : $O - A_3$.

and the angle α is defined by $\cotan(\alpha) = 2$. We remind the expression of the smooth approximation $\tilde{F} = d$ for the \min function, in the third quadrant:

$$\tilde{F}(P) = d = \begin{cases} -\frac{-b \pm (b^2 - 4ac)^{0.5}}{2a} & \text{if } a \neq 0 \text{ and } P \text{ in zone E} \\ \frac{c}{b} & \text{if } a = 0 \text{ and } P \text{ in zone E} \\ y & \text{if } P \text{ in zone F} \\ x & \text{if } P \text{ in zone D} \end{cases}$$

where $a = \tan^2(\alpha) + 2 \tan(\alpha) - 1 = \frac{1}{4}$, $b = 2(x + y) \tan(\alpha) = (x + y)$ and $c = x^2 + y^2$.

Zone IV In zone IV, we switch to a fixed radius approximation with the bounding band. Outside the bounding band, we take $\tilde{F} = \min(x, y)$. Inside the bounding band, we start from the equation of the circle: $(x - x_0)^2 + (y - y_0)^2 = R^2$, where x_0, y_0 are parameters that can be expressed using the radius R and the distance d , which is the searched value of \tilde{F} .

Fig. 15 shows these different parameters and their geometric relations. It is obvious that $|x_0| = |y_0| = d + R$. Replacing x_0 and y_0 in the equation of the circle, the following quadratic expression of the unknown d is obtained: $2d^2 + d(2x + 2y - 4R) + (x^2 + y^2 - 2xR - 2yR + R^2) = 0$. The solution for the unknown d gives the value $d = -\frac{-b \pm (b^2 - 4ac)^{0.5}}{2a}$, where $a = 2$, $b = 2x + 2y - 4R$ and $c = x^2 + y^2 - 2xR - 2yR + R^2$ are the coefficients of the quadratic equation.

The expression for \tilde{F} in zone IV becomes:

$$\tilde{F}(P) = d = \begin{cases} -\frac{-b \pm (b^2 - 4ac)^{0.5}}{2a} & \text{inside the bounding band} \\ y & \text{below } y = x - R \\ x & \text{above } y = x + R \end{cases}$$

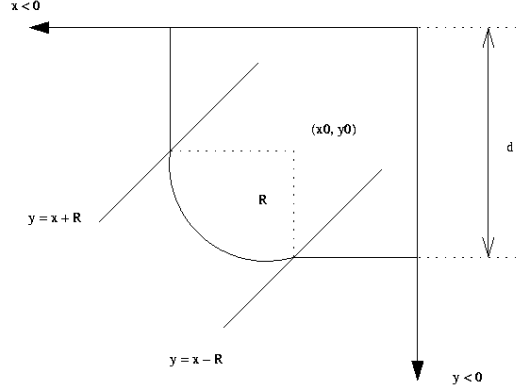


Figure 15: Geometric relations between the parameters of the circular approximation with the bounding band in zone IV.

where $a = 2$, $b = 2x + 2y - 4R$ and $c = x^2 + y^2 - 2xR - 2yR + R^2$.

Final expression for the SARDF intersection in the quadrant III We give the final expression for the SARDF intersection \tilde{F} approximating the *min* function in the third quadrant. Given a point $P(x, y)$ in the third quadrant, the value d of \tilde{F} at P is defined as:

$$\tilde{F}(P) = d = \begin{cases} -\frac{b_1 \pm (b_1^2 - 4a_1c_1)^{0.5}}{2a_1} & \text{if } a_1 \neq 0 \text{ and } P \text{ in zone III, E} \\ \frac{c_1}{b_1} & \text{if } a_1 = 0 \text{ and } P \text{ in zone III, E} \\ y & \text{if } P \text{ in zone III, F} \\ x & \text{if } P \text{ in zone III, D} \\ -\frac{b_2 \pm (b_2^2 - 4a_2c_2)^{0.5}}{2a_2} & \text{if } P \text{ in zone IV, and inside the bounding band} \\ y & \text{if } P \text{ in zone IV and below } y = x - R \\ x & \text{if } P \text{ in zone IV and above } y = x + R \end{cases}$$

where $a_1 = \tan^2(\alpha) + 2 \tan(\alpha) - 1 = \frac{1}{4}$, $b_1 = 2(x + y) \tan(\alpha) = (x + y)$, $c_1 = x^2 + y^2$, $a_2 = 2$, $b_2 = 2x + 2y - 4R$ and $c_2 = x^2 + y^2 - 2xR - 2yR + R^2$.

3.2 SARDF union

3.2.1 Approximate *max*: quadrant I

The intersections of the two parallel bounding band lines, $y = x + R$ and $y = x - R$, with the lines parallel to the axes, $x = R$ and $y = R$, result in two points: $A_1(2R, R)$ and $A_2(R, 2R)$, as shown in Fig. 16. These points are connected by the circular arc $(x - R)^2 + (y - R)^2 = R^2$. This makes a natural boundary that splits the first quarter into two zones I and II, for applying two different approaches to approximation (see Fig. 16).

Zone I Zone I corresponds to the set of points $P(x, y)$ for which $x < R$ or $y < R$ or $(x < 2R$ and $y < 2R$ and $(x - R)^2 + (y - R)^2 < R^2$), see Fig. 16. In this zone, the circular approximation described in 2.2.1 is used. In this case, the bounding lines for the angle θ are:

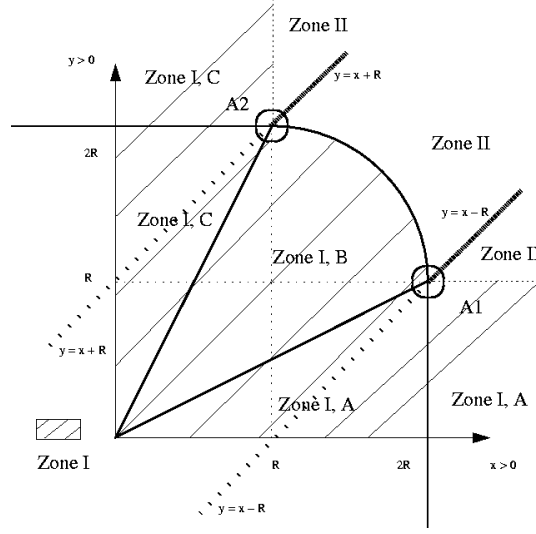


Figure 16: The first quadrant is divided into two zones. The circular approximation is applied in zone I, the fixed radius approximation with the bounding band is applied in zone II.

- L_1 : $O - A_1$, where O is the origin $(0,0)$,
- L_2 : $O - A_2$.

and the angle α is defined by $\cotan(\alpha) = 2$. We remind the expression of the circular approximation $\tilde{G} = d$ for the \max function in the first quadrant:

$$\tilde{G}(P) = d = \begin{cases} \frac{-b \pm (b^2 - 4ac)^{0.5}}{2a} & \text{if } a \neq 0 \text{ and P in zone B} \\ -\frac{c}{b} & \text{if } a = 0 \text{ and P in zone B} \\ x & \text{if P in zone A} \\ y & \text{if P in zone C} \end{cases}$$

where $a = \tan^2(\alpha) + 2 \tan(\alpha) - 1 = \frac{1}{4}$, $b = -2(x + y) \tan(\alpha) = -(x + y)$ and $c = x^2 + y^2$.

Zone II In zone II, the fixed radius with the bounding band is applied to get the approximate function \tilde{G} of the \max function. Outside the bounding band lines $y = x + R$ and $y = x - R$, we take $\tilde{G} = \max(x, y)$. Inside the bounding band, we start from the equation of the circle: $(x - x_0)^2 + (y - y_0)^2 = R^2$, where x_0, y_0 are parameters that can be expressed using the radius R and the distance d , which is the searched value of \tilde{G} .

Fig. 17 displays these different parameters and their geometric relations. It is obvious that $x_0 = y_0 = d - R$. Replacing x_0 and y_0 in the equation of the circle, the following quadratic expression of the unknown d is obtained: $2d^2 + d(-2x - 2y - 4R) + (x^2 + y^2 + 2xR + 2yR + R^2) = 0$. The solution for the unknown d , gives the searched value for the smooth approximation to \max : $d = \frac{-b \pm (b^2 - 4ac)^{0.5}}{2a}$, with $a = 2$, $b = -2x - 2y - 4R$ and $c = x^2 + y^2 + 2xR + 2yR + R^2$ being the coefficients of the quadratic equation.

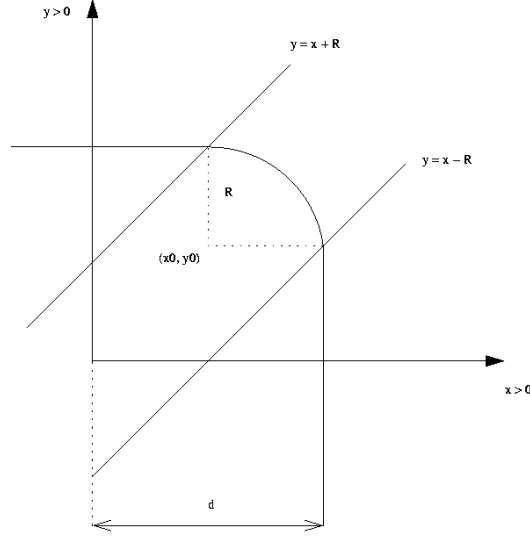


Figure 17: Geometric relations between the parameters of the circular approximation with the bounding band in zone II.

The expression for \tilde{G} in the zone II becomes:

$$\tilde{G}(P) = d = \begin{cases} \frac{-b \pm (b^2 - 4ac)^{0.5}}{2a} & \text{inside the bounding band} \\ x & \text{below } y = x - R \\ y & \text{above } y = x + R \end{cases}$$

where $a = 2$, $b = -2x - 2y - 4R$ and $c = x^2 + y^2 + 2xR + 2yR + R^2$.

Final expression for the SARDF union in the quadrant I We give the final expression for the approximation \tilde{G} of the *max* function in the first quadrant. Given a point $P(x, y)$ in the first quadrant, the value d of \tilde{G} at P is:

$$\tilde{G}(P) = d = \begin{cases} \frac{-b_1 \pm (b_1^2 - 4a_1c_1)^{0.5}}{2a_1} & \text{if } a_1 \neq 0 \text{ and } P \text{ in zone I, B} \\ -\frac{c_1}{b_1} & \text{if } a_1 = 0 \text{ and } P \text{ in zone I, B} \\ x & \text{if } P \text{ in zone I, A} \\ y & \text{if } P \text{ in zone I, C} \\ \frac{-b_2 \pm (b_2^2 - 4a_2c_2)^{0.5}}{2a_2} & \text{if } P \text{ in zone II, and inside the bounding band} \\ x & \text{if } P \text{ in zone II and below } y = x - R \\ y & \text{if } P \text{ in zone II and above } y = x + R \end{cases}$$

where $a_1 = \tan^2(\alpha) + 2 \tan(\alpha) - 1 = \frac{1}{4}$, $b_1 = -2(x + y) \tan(\alpha) = -(x + y)$, $c_1 = x^2 + y^2$, $a_2 = 2$, $b_2 = -2x - 2y - 4R$ and $c_2 = x^2 + y^2 + 2xR + 2yR + R^2$.

3.2.2 Approximate *max*: quadrant III

We apply here the approximation of the bounding band to the *max* function in the third quadrant. The intersections of the bounding band lines, $y = x + R$

and $y = x - R$, with the lines parallel to the axes, $x = -R$ and $y = -R$, result in two points: $A_3(-2R, -R)$ and $A_4(-R, -R)$, as shown in Fig. 18. These points are connected by the circular arc $(x + 2R)^2 + (y + 2R)^2 = R^2$. This makes a natural boundary, that splits the third quarter into two zones III and IV, for applying two different approaches to approximation (see Fig. 18).

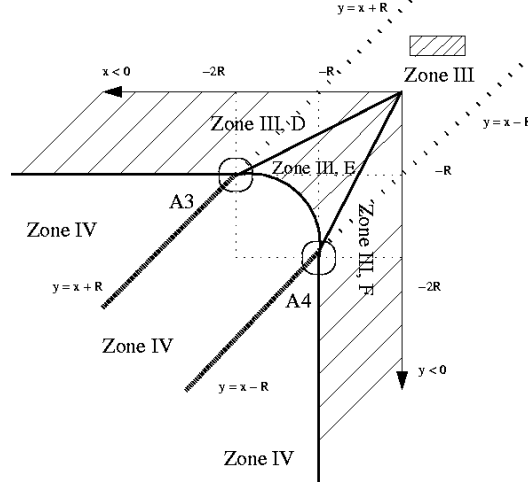


Figure 18: The first quadrant is divided into two zones. The circular approximation is applied in zone III, the fixed radius approximation with the bounding band is applied in zone IV.

Zone III Zone III corresponds to the set of points $P(x, y)$ for which $x > -R$ or $y > -R$ or $(x > -2R$ and $y > -2R$ and $(x + 2R)^2 + (y + 2R)^2 > R^2$), see Fig. 18. In this zone, the circular approximation described in 2.2.2 is applied. In this case, the bounding band lines are:

- L_1 : $O - A_4$, where O is the origin $(0,0)$,
- L_2 : $O - A_3$.

and the angle α is defined by $\cotan(\alpha) = 2$. We remind the expression of the circular approximation $\tilde{G} = d$ for the \max function, in the third quadrant:

$$\tilde{G}(P) = d = \begin{cases} -\frac{-b \pm (b^2 - 4ac)^{0.5}}{2a} & \text{if } a \neq 0 \text{ and } P \text{ in zone E} \\ \frac{c}{b} & \text{if } a = 0 \text{ and } P \text{ in zone E} \\ y & \text{if } P \text{ in zone D} \\ x & \text{if } P \text{ in zone F} \end{cases}$$

where $a = \cotan^2(\alpha) + 2 \cotan(\alpha) - 1 = 7$, $b = 2(x + y) \cotan(\alpha) = 4(x + y)$ and $c = x^2 + y^2$.

Zone IV In zone IV, we switch to the fixed radius approximation with the bounding band in order to get the approximate function \tilde{G} of the \max function. Outside the bounding band lines $y = x + R$ and $y = x - R$, we take $\tilde{G} =$

$\max(x, y)$. Inside the bounding band, we start from the equation of the circle: $(x - x_0)^2 + (y - y_0)^2 = R^2$, where x_0, y_0 are parameters that can be expressed using the radius R and the distance d , the searched value of \tilde{G} .

Fig. 19 displays these different parameters and their geometric relations. It is obvious that $|x_0| = |y_0| = d + R$. Replacing x_0 and y_0 in the equation of the circle, the following quadratic expression of the unknown d is obtained: $2d^2 + d(2x + 2y + 4R) + (x^2 + y^2 + 2xR + 2yR + R^2) = 0$. The solution for the unknown d , gives the searched value for the approximation: $d = -\frac{-b \pm (b^2 - 4ac)^{0.5}}{2a}$, with $a = 2$, $b = 2x + 2y + 4R$ and $c = x^2 + y^2 + 2xR + 2yR + R^2$ being the coefficients of the quadratic equation.

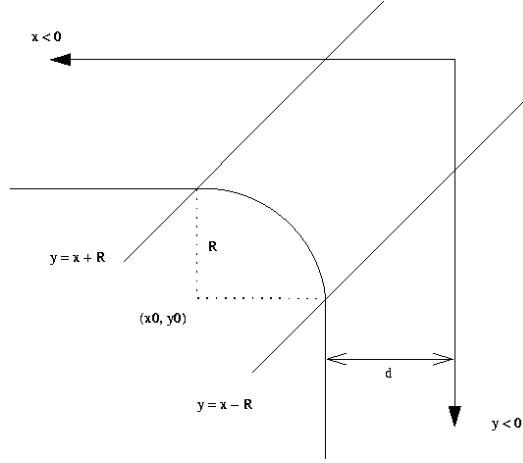


Figure 19: Geometric relations between the parameters of the circular approximation with the bounding band in zone IV.

The expression for \tilde{G} in the zone IV becomes:

$$\tilde{G}(P) = d = \begin{cases} -\frac{-b \pm (b^2 - 4ac)^{0.5}}{2a} & \text{inside the bounding band} \\ x & \text{below } y = x - R \\ y & \text{above } y = x + R \end{cases}$$

where $a = 2$, $b = 2x + 2y + 4R$ and $c = x^2 + y^2 + 2xR + 2yR + R^2$.

Final expression for SARDF union in the quadrant III We give the final expression for the approximation \tilde{G} of the \max function in the third quadrant. Given a point $P(x, y)$ in the third quadrant, the value d of \tilde{G} at P is given by:

$$\tilde{G}(P) = d = \begin{cases} -\frac{-b_1 \pm (b_1^2 - 4a_1c_1)^{0.5}}{2a_1} & \text{if } a_1 \neq 0 \text{ and } P \text{ in zone III, E} \\ \frac{c_1}{b_1} & \text{if } a_1 = 0 \text{ and } P \text{ in zone III, E} \\ x & \text{if } P \text{ in zone III, F} \\ y & \text{if } P \text{ in zone III, D} \\ -\frac{-b_2 \pm (b_2^2 - 4a_2c_2)^{0.5}}{2a_2} & \text{if } P \text{ in zone IV, and inside the bounding band} \\ x & \text{if } P \text{ in zone IV and below } y = x - R \\ y & \text{if } P \text{ in zone IV and above } y = x + R \end{cases}$$

where $a_1 = \cotan^2(\alpha) + 2 \cotan(\alpha) - 1 = 7$, $b_1 = 2(x+y) \cotan(\alpha) = 4(x+y)$, $c_1 = x^2 + y^2$, $a_2 = 2$, $b_2 = 2x + 2y - 4R$ and $c_2 = x^2 + y^2 + 2xR + 2yR + R^2$.

4 Examples

4.1 Intersection and union of two half planes

We present the results of applying the proposed SARDF intersection and union (C^1 continuous approximations of the *min* and *max* functions) of the half planes $f_1 = x, f_1 \geq 0$ and $f_2 = y, f_2 \geq 0$ (see Fig. 20 bottom and Fig. 21 bottom). We also present for comparison purposes the contour maps for the *min* and *max* approaches (see Fig. 20 top left and Fig. 21 top left) and for the R-Function intersection and union (see Fig. 20 top right and Fig. 21 top right).

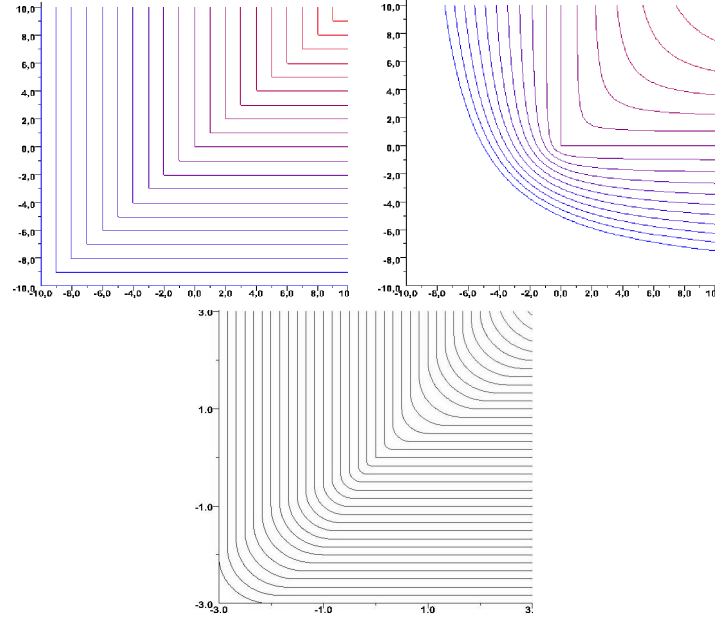


Figure 20: Contour lines of (Top left) the *min* function, (Top right) the R-Function intersection and (Bottom) the SARDF intersection, of two half spaces $f_1 = x, f_1 \geq 0$ and $f_2 = y, f_2 \geq 0$.

It is easy to see that both in the case of the SARDF intersection (see Fig. 20 (bottom)) and union (see Fig. 21 (bottom)), the circular arc shape is growing, until reaching the given threshold for the radius, then stops growing and is confined inside the bounding band.

The contour lines for the intersection and the union of the two half spaces using the *min* and *max* functions (see Fig. 20 (top left) and Fig. 21 (top left)) show the C^1 discontinuity in each contour lines, whereas the SARDF approach (see Fig. 20 bottom and Fig. 21 bottom) has a C^1 discontinuity only in the origin.

The contour lines for the intersection and the union of the two half spaces

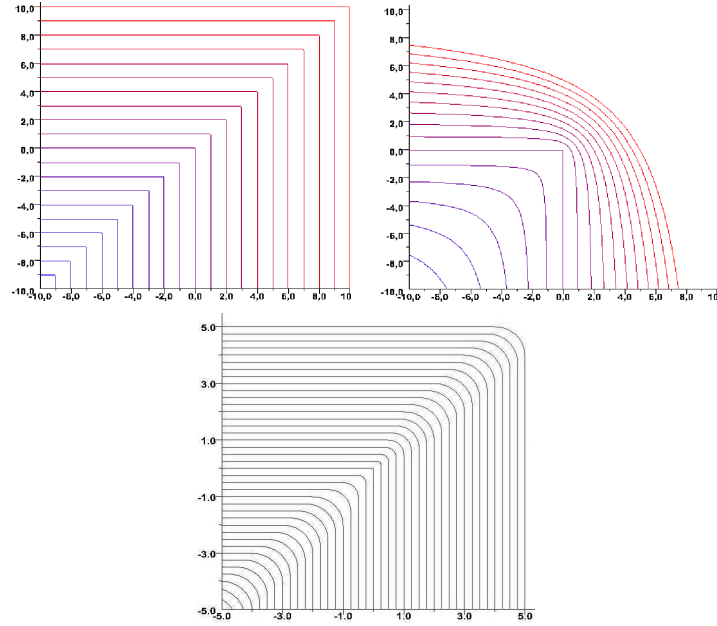


Figure 21: Contour lines of (Top left) the \max function, (Top right) the R-Function union and (Bottom) the SARDF union, of two half spaces $f_1 = x$, $f_1 \geq 0$ and $f_2 = y$, $f_2 \geq 0$.

using the R-Function intersection and union (see Fig. 20 (top right) and Fig. 21 (top right)) show that such functions do not provide distance values (neither exact nor approximate to some bounded error) and suffer from an exponential function value growth.

4.2 Examples with two disks

We apply the SARDF operators to describe union, intersection and difference of two disks in the plane. Note that the difference between f_1 and f_2 is obtained by the SARDF intersection of f_1 and $-f_2$.

The union is illustrated by the contour map of the resulting function in Fig. 22, the intersection is illustrated by Fig. 24, and the difference by Fig. 23. See how in each case, the sharp feature is conserved for the 0 isovalue, whereas the other contours are smoothed by the proposed approximation.

5 Estimation of the distance error for SARDF operations

5.1 Upper limit of the distance error for a single SARDF operation

The upper limit of the distance error for a single SARDF operation is reached in the band area, where the traditional operation (\min/\max) is replaced by an

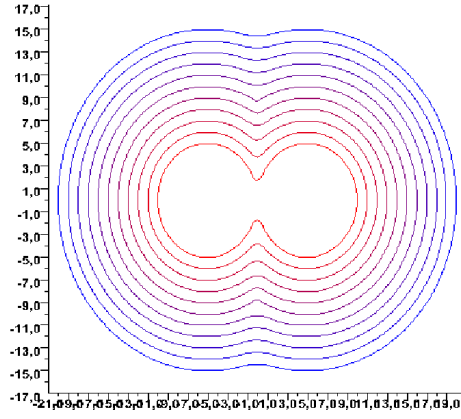


Figure 22: Contour map of SARDF union of two disks.

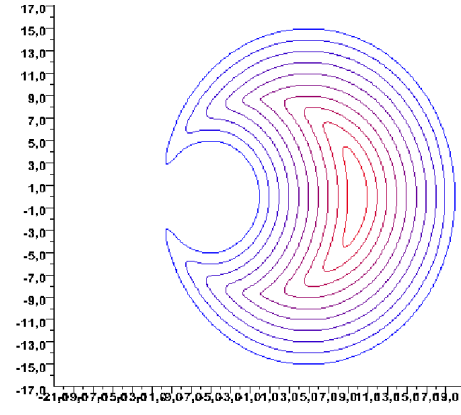


Figure 23: Contour map of SARDF difference of two disks.

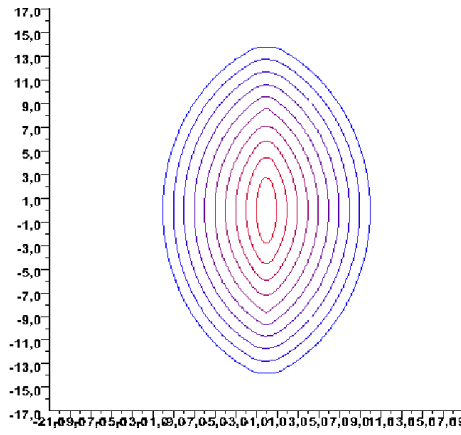


Figure 24: Contour map of SARDF intersection of two disks.

arc of circle of fixed radius R . For reason of symmetry, the upper limit error for the SARDF intersection and union is the same in absolute value in all the quadrants, so only one case needs to be studied. Let us consider the SARDF intersection in the first quadrant. Fig. 25 reminds the configuration of the first quadrant, with the smooth approximation and the different zones.

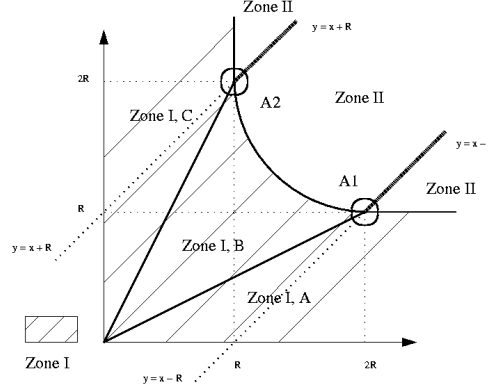


Figure 25: The first quadrant is divided into two zones. The circular approximation is applied in zone I, whereas we introduce a fixed radius approximation with the bounding band in zone II.

The error made at one given point (x, y) , when using the SARDF intersection instead of the *min* itself in the circular shape area (it can be both in the Zone I,B or in the zone II) is shown in Fig. 26. Let (x, y) be one point for which the distance approximation to the intersection of the two half-planes $x \geq 0$ and $y \geq 0$ is computed. Outside the angle zone used for the circular approximation (zone I,B in Fig. 25) and outside the bounding band (zone II, in Fig. 25), the error is 0. An error is introduced only in the zones of the smooth approximation: zone I,B and zone II (Fig. 25). One upper limit error can be computed in the zone II, inside the band. Fig. 27 shows this area, with the distance error at different points in dashed line. The upper limit for the distance error is reached at the point P defined by $x = y$.

For that particular point, the error is exactly $\epsilon = y = x$; $P(x, y)$ being a point of the circular arc shape of radius R , its coordinates follow the equation: $(x - R)^2 + (y - R)^2 = R^2$. Since $x = y$, it follows that: $2x^2 - 4Rx + R^2 = 0$, for which the two solutions can be easily obtained. One of this solution can be discarded, since it does not respect the condition $x \leq R$, therefore only $x = \frac{(\sqrt{2}-1)}{\sqrt{2}}R$ remains. This value is an upper bound for the distance error made when using SARDF.

In the case of the SARDF union operator, the absolute value of the error should be subtracted to the computed approximate distance, in order to have the correct distance value; whereas in the case of the SARDF intersection operator, the absolute value of the error needs to be added to the computed approximate distance in order to get the correct distance value.

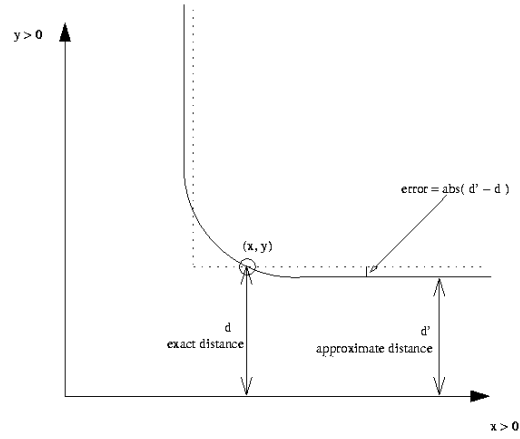


Figure 26: Exact distance, computed approximate distance and error at a given point (x, y) .

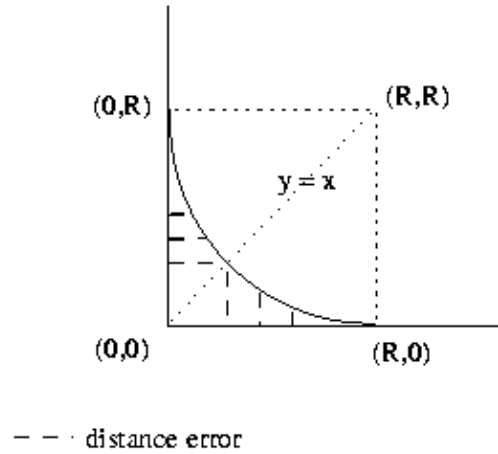


Figure 27: Distance error (in dashed lines) at different points in the band zone (zone II), and max error reached for $x = y$.

5.2 Algorithm for the exact distance error evaluation

The distance error can be evaluated exactly at any point with the following steps:

1. Evaluate the approximate distance using SARDF operations in the constructive tree nodes;
2. Replace in the constructive tree all the occurrences of the SARDF intersection and union, by *min* and *max* functions;
3. Evaluate the exact distance function by traversing the tree with the replaced nodes;
4. The distance error is the difference between the two above computed values.

Note that the distance error can be zero depending on the given point position and the operations applied in the construction.

6 Comparison of time efficiency between SARDF and the *min/max* operators

Because the SARDF intersection and union are more complicated than the *min* and *max* functions, they require more time for their execution. Therefore, we looked at their time efficiency and checked that the overhead in time, compared to the *min/max* functions but also to the R-Function intersection and union, remains reasonable and does not forbid a practical use of these functions.

Subdivisions	Time (in sec.)					
	SARDF intersection	<i>min</i>	R-Function intersection	SARDF union	<i>max</i>	R-Function union
1001 * 1001	0.05	0.01	0.02	0.05	0.01	0.02
10001 * 10001	5.29	1.42	2.52	4.97	1.38	2.56
20001 * 20001	21.18	5.65	10.09	19.59	5.53	10.07
30001 * 30001	47.47	12.66	22.75	44.07	12.46	22.72

Table 1: Time efficiency for 1001*1001, 10001*10001, 20001*20001 and 30001*30001 evaluations of SARDF intersection, SARDF union, *min*, *max*, R-Function for the intersection and R-Function for the union.

In order to measure the time efficiency, a square was considered in the $x - y$ plane, starting from $(-10, -10)$ and going to $(10, 10)$; this square is regularly subdivided along the x and y axis; let n_x and n_y be the number of subdivisions along the x and y axis respectively. Then, for each of the functions, the time taken for evaluating these $n_x \times n_y$ points in the square is considered. The results of these evaluations for different subdivisions and all the functions (SARDF intersection and union, *min* and *max*, R-Function intersection and union) is given in Table 1.

The overhead in time between the SARDF intersection, *min* and the R-Function intersection in one hand, and between the SARDF union, *max* and the R-Function intersection in the other hand are shown in Fig. 28 and Fig. 29 respectively. In our test, we found a factor of approximately 4 between the SARDF operators and the traditional *min/max*, and a factor of approximately 2 between the SARDF operators and the R-Function.

All the functions considered in the tests were implemented in ANSI C and compiled with the optimization flags turned on, the time results were obtained on a Pentium 4 processor, with 256 MBytes of RAM.

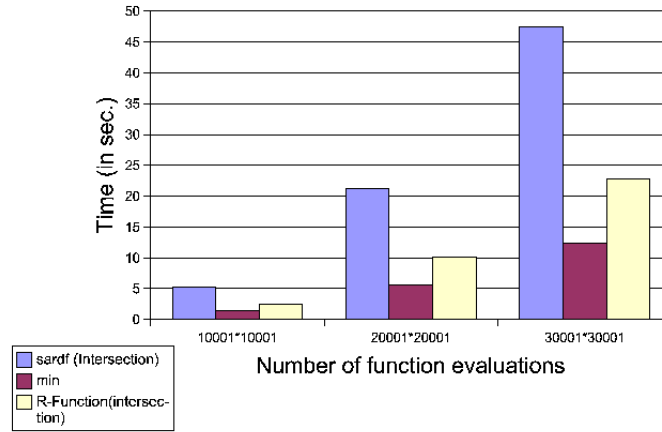


Figure 28: Time comparison between SARDF intersection, *min* and R-Function intersection for different number of function evaluations.

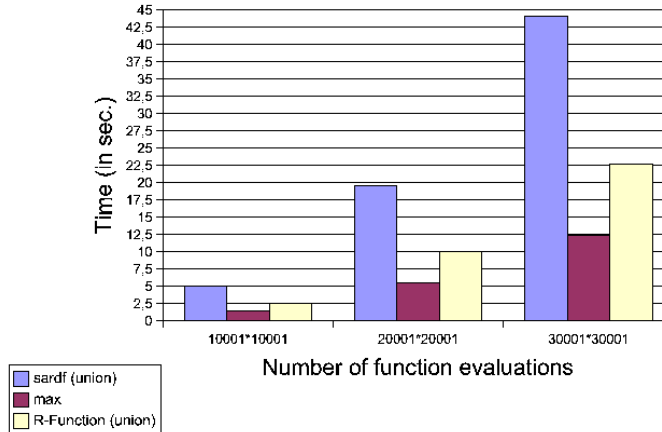


Figure 29: Time comparison between SARDF union, *max* and R-Function union for different number of function evaluations.

7 Conclusion

In this document, we provide a new set of functions defining set-theoretic operations for constructive modeling. Under the condition that the primitives being used for modeling are defined by exact distance functions, the proposed set of functions provides for a good approximation of the real distance value. Furthermore, the upper boundary of the error in the distance calculation can be exactly determined.

The proposed operations are first defined for two half planes as $x \geq 0$ and as $y \geq 0$ in each quadrant of a two-dimensional space. While in quadrants II and IV they are equal to *min/max* functions, in quadrants I and III we use a circular *min/max* approximation with an additional bounding band to control the distance function error. The resulting signed approximate real distance functions (SARDF) are at least C^1 continuous, with the exception of the point $(0,0)$.

The SARDF operations have several applications, such as rapid ray-tracing, metamorphosis, blending and others. In our future research, we will apply the proposed functions to heterogeneous object modeling. A heterogeneous object is an object composed of several different materials with variable distribution. The SARDF provide a real approximate distance, and thus should allow one first to define easily the distribution of a given material in an object, but also to control it precisely.

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